

Extended Kerr–Schild spacetimes: General properties and some explicit examples

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Abstract. We study extended Kerr–Schild (xKS) spacetimes, i.e. an extension of the Kerr–Schild (KS) ansatz where, in addition to the null KS vector, a spacelike vector field appears in the metric. In contrast to the KS case, we obtain only a necessary condition under which the KS vector is geodesic. However, it is shown that this condition becomes sufficient if we appropriately restrict the geometry of the null KS and spacelike vector. It turns out that xKS spacetimes with a geodesic KS vector are of Weyl type I and the KS vector has the same optical properties in the full and background spacetimes. In the case of Kundt xKS spacetimes, the compatible Weyl types are further restricted and a few examples of such metrics are provided. The relation of pp -waves to the class of Kundt xKS spacetimes is briefly discussed. We also show that the CCLP black hole, as an example of an expanding xKS spacetime, is of Weyl type I_i , which specializes to type D in the uncharged or non-rotating limit, and that its optical matrix satisfies the optical constraint.

1. Introduction

The Einstein field equations constitute a very complex system of partial differential equations of the second order and finding exact solutions is a considerably non-trivial task especially in dimension $n > 4$. One of the possible approaches to this problem is to assume a convenient form for the unknown metric in order to simplify subsequent calculations. An important example representing such a technique is the Kerr–Schild (KS) ansatz [1]

$$g_{ab} = \eta_{ab} - 2\mathcal{H}k_a k_b, \quad (1)$$

where \mathcal{H} is a scalar function and \mathbf{k} is a null vector field with respect to both the Minkowski background metric η_{ab} and the full metric g_{ab} . Despite its simplicity, the class of KS metrics contains many physically important exact solutions of four-dimensional general relativity, such as the Kerr black hole, the charged Kerr–Newman black hole, the Vaidya radiating star, Kinnersley photon rocket, pp -waves and also some of their higher dimensional analogues (see e.g. [2]). In fact, the KS ansatz has led to the discovery of the rotating black holes in higher dimensions [3, 4] with vanishing and non-vanishing cosmological constant, respectively. The KS ansatz has been also successfully applied in

the context of higher order gravities such as the Gauss–Bonnet theory [5,6] or quadratic gravity [7,8].

Various properties of KS spacetimes with flat background and generalized Kerr–Schild (GKS) spacetimes with (anti-)de Sitter backgrounds in arbitrary dimension have been studied in [9] and [10], respectively. It has turned out, for instance, that the GKS metric with a geodetic null congruence \mathbf{k} leads to algebraically special solutions. The aim of this paper is to investigate general properties of an extension of the original KS ansatz referred to as the extended Kerr–Schild (xKS) ansatz that we define as a metric of the form

$$g_{ab} = \bar{g}_{ab} - 2\mathcal{H}k_a k_b - 2\mathcal{K}k_{(a}m_{b)}, \quad (2)$$

where \mathcal{H} , \mathcal{K} are scalar functions, the background metric \bar{g}_{ab} represents a maximally symmetric vacuum, i.e. a Minkowski or (anti-)de Sitter spacetime, \mathbf{k} is a null vector and \mathbf{m} is a spacelike unit vector, both with respect to the full metric

$$k^a k_a \equiv g_{ab}k^a k^b = 0, \quad k^a m_a \equiv g_{ab}k^a m^b = 0, \quad m^a m_a \equiv g_{ab}m^a m^b = 1. \quad (3)$$

It immediately follows from the form of the xKS metric (2) that the same also holds with respect to the background metric

$$\bar{g}_{ab}k^a k^b = 0, \quad \bar{g}_{ab}k^a m^b = 0, \quad \bar{g}_{ab}m^a m^b = 1 \quad (4)$$

and that the inverse metric can be simply expressed as

$$g^{ab} = \bar{g}^{ab} + (2\mathcal{H} - \mathcal{K}^2) k^a k^b + 2\mathcal{K}k^{(a}m^{b)}. \quad (5)$$

Note that our definition of the xKS ansatz (2) slightly differs from those ones in [11,12]. In our notation, the xKS metric (2) reduces to the GKS form [10] for $\mathcal{K} = 0$, moreover, the spacelike vector \mathbf{m} is normalized to unity since we will identify it with one of the frame vectors.

The xKS ansatz has been already studied in [12] through the method of perturbative expansion. It has turned out that the vacuum Einstein field equations truncate beyond a certain low order in the expansion around the flat background metric similarly as in the case of KS spacetimes. In this paper, we employ the higher dimensional generalization of the Newman–Penrose formalism [13,14] and the algebraic classification of the Weyl tensor in higher dimensions based on the existence of preferred null directions and their multiplicity [15,16], see also the recent reviews [17,18]. These tools allow us to formulate some statements about geodeticity and optical properties of the null congruence \mathbf{k} and admissible Weyl types of xKS spacetimes. The results of such analysis could be found helpful in obtaining new exact solutions of the xKS form.

First, let us provide a motivation for studying xKS spacetimes. As will be shown in section 3, one of the reasons why to consider the xKS ansatz (2) is that such metrics cover more general algebraic types than GKS metrics. Recall the results of [10] that GKS spacetimes with a geodetic Kerr–Schild vector \mathbf{k} without any further assumptions are of Weyl type II. Expanding Einstein GKS spacetimes are compatible only with type D or genuine type II unless conformally flat. Non-expanding Einstein GKS spacetimes are of

type N and belong to the Kundt class. Therefore, expanding Einstein xKS spacetimes could include black hole solutions of more general Weyl types than II, for instance black rings [19] that are of type I_i [20], and non-expanding Einstein xKS spacetimes do indeed contain Kundt metrics of more general types than N as will be shown in section 4.

Furthermore, unlike the static charged black hole, rotating charged black hole as an exact solution of higher dimensional Einstein–Maxwell theory is unknown. The four-dimensional Kerr–Newman black hole can be cast to the KS form with the flat background metric η_{ab} , the function \mathcal{H} , and the Kerr–Schild vector \mathbf{k} given by [21]

$$\eta_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2, \quad (6)$$

$$\mathcal{H} = -\frac{r^2}{r^4 + a^2 z^2} \left(Mr - \frac{Q^2}{2} \right), \quad (7)$$

$$k_a dx^a = dt + \frac{rx + ay}{r^2 + a^2} dx + \frac{ry - ax}{r^2 + a^2} dy + \frac{z}{r} dz, \quad (8)$$

where M , Q are mass and charge of the black hole, respectively. The coordinate r satisfies

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \quad (9)$$

and the vector potential is proportional to the Kerr–Schild vector \mathbf{k}

$$A = \frac{Qr^3}{r^4 + a^2 z^2} \mathbf{k}. \quad (10)$$

However, the attempt to generalize this solution to higher dimensions using the KS form of the Myers–Perry black hole has failed [3]. It has turned out that a vector potential proportional to \mathbf{k} cannot simultaneously satisfy the corresponding Einstein and Maxwell field equations. It is also known that a straightforward generalization of five-dimensional rotating black hole solutions of general relativity in the GKS form to the Gauss–Bonnet theory [5] does not represent rotating black holes [6] and therefore such spacetimes were so far studied only numerically [22, 23] or in the limit of small angular momentum [24]. An open question is whether the xKS ansatz may resolve the above-mentioned problems and lead to these so far unknown exact solutions.

Another significant reason for studying xKS spacetimes is that some of the known exact solutions can be cast to the xKS form. For instance, metrics with vanishing scalar invariants (VSI) [25, 26] are examples of xKS spacetimes as will be pointed out in section 4. Some expanding xKS spacetimes representing black holes are also known, namely the Chong, Cvetič, Lü, and Pope (CCLP) solution [27] discussed in section 5. In fact, the investigation of the CCLP metric in [11] has led to the introduction of the notion of the xKS form.

Lastly, the xKS metric seems to be still sufficiently simple to be treated analytically with occasional help of the computer algebra system CADABRA [28, 29] which allows to perform otherwise tedious symbolic calculations in arbitrary unspecified dimension.

The paper is organized as follows. In section 2, we analyze xKS spacetimes with a general null Kerr–Schild vector field \mathbf{k} , compare frames in the full and background metrics and discuss the relation of the vectors \mathbf{k} and \mathbf{m} appearing in the xKS ansatz. In

section 3, we assume \mathbf{k} to be geodetic which allows us to express the Ricci and Riemann tensors explicitly and, consequently, determine algebraic type of xKS spacetimes. Kundt xKS metrics are examined in section 4. This assumption on the geometry of the KS vector \mathbf{k} simplifies the subsequent calculations and the possible Weyl type further specializes depending on the form of the scalar function \mathcal{K} and the Ricci tensor. We also provide examples of Kundt spacetimes admitting the xKS form and discuss the relation of Ricci-flat pp -waves to the xKS class. Finally, in section 5, the CCLP black hole as an example of an expanding xKS spacetime is studied. For this solution, we establish a null frame, determine the Weyl type and show that the optical matrix obeys the optical constraint.

1.1. Preliminaries

Throughout the paper, we assume that the dimension $n \geq 4$ and that the background metric \bar{g}_{ab} representing an (anti-)de Sitter or Minkowski spacetime takes the conformally flat form

$$\bar{g}_{ab} = \Omega \eta_{ab}, \quad \eta_{ab} dx^a dx^b = -dt^2 + dx_1^2 + \dots + dx_{n-1}^2 \quad (11)$$

with the corresponding conformal factor

$$\Omega_{\text{AdS}} = \frac{(n-2)(n-1)}{2\Lambda t^2}, \quad \Omega_{\text{dS}} = -\frac{(n-2)(n-1)}{2\Lambda x_1^2}, \quad (12)$$

or $\Omega = 1$ in the case of flat background, respectively. The cosmological constant Λ is defined such that the vacuum Einstein field equations read

$$R_{ab} = \frac{2\Lambda}{n-2} g_{ab}. \quad (13)$$

In n -dimensional spacetimes, it is convenient to introduce a real null frame consisting of two null vectors $\mathbf{n} \equiv \mathbf{m}^{(0)}$, $\boldsymbol{\ell} \equiv \mathbf{m}^{(1)}$ and $n-2$ orthonormal spacelike vectors $\mathbf{m}^{(i)}$ obeying

$$n^a n_a = \ell^a \ell_a = n^a m_a^{(i)} = \ell^a m_a^{(i)} = 0, \quad n^a \ell_a = 1, \quad m^{(i)a} m_a^{(j)} = \delta_{ij}. \quad (14)$$

The indices a, b, \dots range from 0 to $n-1$, the indices i, j, \dots running from 2 to $n-1$ number the spacelike frame vectors. Under boosts, the frame transforms as

$$\ell' = \lambda \ell, \quad n' = \lambda^{-1} n, \quad m'^{(i)} = m^{(i)} \quad (15)$$

and we say that some quantity q has a boost weight w if it obeys

$$q' = \lambda^w q. \quad (16)$$

We adopt the notation of the higher dimensional Newman–Penrose formalism [13, 14, 25]. Namely, the Ricci rotation coefficients L_{ab} , N_{ab} and $\overset{i}{M}_{bc}$ are defined as frame components of covariant derivatives of the frame vectors

$$\ell_{a;b} = L_{cd} m_a^{(c)} m_b^{(d)}, \quad n_{a;b} = N_{cd} m_a^{(c)} m_b^{(d)}, \quad m_{a;b}^{(i)} = \overset{i}{M}_{cd} m_a^{(c)} m_b^{(d)} \quad (17)$$

and the directional derivatives along the corresponding frame vectors are denoted as

$$D \equiv \ell^a \nabla_a, \quad \Delta \equiv n^a \nabla_a, \quad \delta_i \equiv m_{(i)}^a \nabla_a. \quad (18)$$

The optical matrix L_{ij} has a special geometrical meaning. If ℓ is geodetic, L_{ij} is invariant under null rotations with ℓ fixed and can be decomposed as

$$L_{ij} = S_{ij} + A_{ij}, \quad S_{ij} \equiv L_{(ij)} = \sigma_{ij} + \theta \delta_{ij}, \quad A_{ij} \equiv L_{[ij]}, \quad (19)$$

where the trace θ , symmetric traceless part σ_{ij} , and anti-symmetric part A_{ij} are related to the expansion, shear, and twist of the geodetic congruence ℓ , respectively. The shear and twist scalars are defined as

$$\sigma^2 \equiv \sigma_{ij} \sigma_{ij}, \quad \omega^2 \equiv A_{ij} A_{ij}. \quad (20)$$

According to the form of the xKS ansatz, it is convenient to identify the null and spacelike vectors \mathbf{k} , \mathbf{m} appearing in the xKS metric (2) with the vectors ℓ , $\mathbf{m}^{(2)}$ of the null frame (14), respectively. The corresponding Ricci rotation coefficients will be denoted as L_{ab} and $M_{ab} \equiv \overset{2}{M}_{ab}$. It is also useful to define indices $\tilde{i}, \tilde{j}, \dots = 3, \dots, n-1$ denoted by tilde such that the vector $\mathbf{m} \equiv \mathbf{m}^{(2)}$ is excluded in the notation $\mathbf{m}^{(\tilde{i})}$.

2. General Kerr–Schild vector field

As in the case of the GKS ansatz [10], first important result follows from the boost weight 2 component of the Ricci tensor $R_{00} = R_{ab} k^a k^b$. This is the simplest component since \mathbf{k} and \mathbf{m} obey (3) and thus many terms of the Ricci tensor of the xKS metric vanish when twice contracted with the null vector \mathbf{k} . Although the calculations are much more involved than in the case of GKS spacetimes, we obtain a quite simple expression

$$\begin{aligned} R_{00} = & 2\mathcal{H}L_{i0}L_{i0} - \frac{1}{2}\mathcal{K}^2 L_{\tilde{i}0}L_{\tilde{i}0} + 2\mathcal{K}L_{i(i}L_{2)0} + \mathcal{K}L_{\tilde{i}0}M_{\tilde{i}0} + 2D\mathcal{K}L_{20} + \mathcal{K}DL_{20} \\ & - \frac{1}{2}(n-2) \left(\frac{\Omega_{,ab}}{\Omega} - \frac{3}{2} \frac{\Omega_{,a}\Omega_{,b}}{\Omega^2} \right) k^a k^b. \end{aligned} \quad (21)$$

The last term vanishes identically since we assume the background metric \bar{g}_{ab} to be a maximally symmetric vacuum for which the conformal factor is given by (12) or $\Omega = 1$ in the case of (A)dS or Minkowski spacetime, respectively. Obviously, the component R_{00} vanishes completely if $L_{i0} = 0$ and therefore

Proposition 1 *The boost weight 2 component of the Ricci tensor $R_{00} = R_{ab} k^a k^b$ vanishes if the null vector field \mathbf{k} in the extended Kerr–Schild metric (2) is geodetic.*

In the context of general relativity, proposition 1 implies that for a geodetic \mathbf{k} the boost weight 2 component of the energy–momentum tensor vanishes $T_{00} = 0$ which holds not only if the energy–momentum tensor is absent, i.e. for Einstein spaces, but also for spacetimes containing matter fields aligned with the Kerr–Schild vector \mathbf{k} such as aligned Maxwell field $F_{ab} k^b \propto k_a$ or aligned null radiation $T_{ab} \propto k_a k_b$. Note that in case $\mathcal{K} = 0$, the xKS ansatz (2) reduces to the GKS form and the implication of proposition 1 becomes an equivalence [10]. For $\mathcal{K} \neq 0$, the condition is only sufficient and thus R_{00} vanishes also for a non-geodetic \mathbf{k} with a special choice of \mathcal{H} , \mathcal{K} and \mathbf{m} .

2.1. Kerr–Schild congruence in the background

One may easily relate the null frames in the full and background spacetimes. Starting with the full metric g_{ab} expressed in terms of the null frame (14) as

$$g_{ab} = 2k_{(a}n_{b)} + m_a m_b + \delta_{ij} m_a^{(i)} m_b^{(j)}, \quad (22)$$

we compare this form with the xKS ansatz (2). Obviously, if we set

$$\bar{n}_a = n_a + \mathcal{H}k_a + \mathcal{K}m_a, \quad (23)$$

then $\bar{g}_{ab} = 2k_{(a}\bar{n}_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}$ and thus $\bar{\mathbf{n}}, \mathbf{k}, \mathbf{m}, \mathbf{m}^{(\bar{i})}$ form a null frame in the background spacetime. The indices of the vectors \mathbf{k} and $\mathbf{m}^{(\bar{i})}$ can be raised and lowered by both metrics g_{ab} and \bar{g}_{ab} , however, one has to treat the vectors $\bar{\mathbf{n}}$ and \mathbf{m} carefully since

$$\bar{n}^a \equiv g^{ab} \bar{n}_b = n^a + \mathcal{H}k^a + \mathcal{K}m^a, \quad m^a \equiv g^{ab} m_b, \quad (24)$$

while

$$\bar{g}^{ab} \bar{n}_a = n^b - \mathcal{H}k^b, \quad \bar{g}^{ab} m_a = m^b - \mathcal{K}k^b. \quad (25)$$

The covariant derivative compatible with the background metric \bar{g}_{ab} can be simply expressed using the covariant derivative compatible with the full metric g_{ab} and setting $\mathcal{H} = \mathcal{K} = 0$. The relation between the covariant derivatives allows us to compare the Ricci rotation coefficients constructed in the full spacetime using the frame $\mathbf{n}, \mathbf{k}, \mathbf{m}, \mathbf{m}^{(\bar{i})}$ with those ones in the background spacetime denoted by barred letters and expressed in terms of the frame $\bar{\mathbf{n}}, \mathbf{k}, \mathbf{m}, \mathbf{m}^{(\bar{i})}$

$$L_{i0} = \bar{L}_{i0}, \quad L_{10} = \bar{L}_{10}, \quad L_{ij} = \bar{L}_{ij} - \mathcal{K} \delta_{2[i} \bar{L}_{j]0}, \quad (26)$$

$$L_{1i} = \bar{L}_{1i} - \mathcal{H} \bar{L}_{i0} - \frac{1}{2} \mathcal{K} \bar{\Xi}_i + \frac{1}{2} \mathcal{K} \left(\bar{L}_{10} - \frac{D\mathcal{K}}{\mathcal{K}} + \frac{D\Omega}{\Omega} \right) \delta_{2i}, \quad (27)$$

$$L_{i1} = \bar{L}_{i1} - \frac{1}{2} \mathcal{K} \bar{\Xi}_i - \frac{1}{2} \mathcal{K} \left(\bar{L}_{10} + \frac{D\mathcal{K}}{\mathcal{K}} - \frac{D\Omega}{\Omega} \right) \delta_{2i}, \quad (28)$$

$$L_{11} = \bar{L}_{11} - \mathcal{H} \bar{L}_{10} - D\mathcal{H} - \mathcal{K} \bar{\Theta} + \left(\mathcal{H} - \frac{1}{2} \mathcal{K}^2 \right) \frac{D\Omega}{\Omega}, \quad (29)$$

$$M_{i0} = \bar{M}_{i0} + \frac{1}{2} \mathcal{K} \bar{L}_{i0}, \quad M_{ij} = \bar{M}_{ij} + \mathcal{K} \bar{L}_{(ij)} + \mathcal{K} (\bar{L}_{[2i]} + \bar{M}_{i0}) \delta_{2j} - \frac{1}{2} \mathcal{K} \frac{D\Omega}{\Omega} \delta_{ij}, \quad (30)$$

$$M_{i1} = \bar{M}_{i1} - \mathcal{H} \bar{L}_{i2} + \left(\mathcal{H} - \frac{1}{2} \mathcal{K}^2 \right) \bar{\Xi}_i - \frac{1}{2} \mathcal{K} (\mathcal{H} \bar{L}_{i0} + 2 \bar{L}_{(1i)} - \bar{M}_{i2}) + \frac{1}{2} \delta_i \mathcal{K}, \quad (31)$$

$$\bar{M}_{j0} = \bar{M}_{j0}, \quad \bar{M}_{j1} = \bar{M}_{j1} + 2\mathcal{H} \bar{L}_{[ij]} + \mathcal{H} \bar{M}_{j0} + \mathcal{K} \bar{M}_{[ij]}, \quad (32)$$

$$\bar{M}_{jk} = \bar{M}_{jk} + \mathcal{K} \left(\bar{L}_{[ij]} + \bar{M}_{j0} \right) \delta_{2k}, \quad N_{i0} = \bar{N}_{i0} - \frac{1}{2} \mathcal{K} \bar{\Xi}_i - \frac{1}{2} (\mathcal{K} \bar{L}_{10} + D\mathcal{K}) \delta_{2i}, \quad (33)$$

$$\begin{aligned} N_{i1} = & \bar{N}_{i1} + \delta_i \mathcal{H} + \mathcal{H} (2 \bar{L}_{1i} - \bar{L}_{i1} + \bar{N}_{i0} - \mathcal{H} \bar{L}_{i0}) - \mathcal{K} (\bar{N}_{2i} + \bar{M}_{i1} + \mathcal{H} \bar{\Xi}_i) - \Delta \mathcal{K} \delta_{2i} \\ & - \left[\mathcal{K} \left(\bar{L}_{11} - \mathcal{H} \bar{L}_{10} - \mathcal{K} \bar{\Theta} - \frac{1}{2} \frac{\Delta \Omega}{\Omega} - \mathcal{K} \frac{\delta_2 \Omega}{\Omega} \right) + \left(\mathcal{H} + \frac{1}{2} \mathcal{K}^2 \right) \frac{D\Omega}{\Omega} \right] \delta_{2i}, \end{aligned} \quad (34)$$

$$N_{ij} = \bar{N}_{ij} + \mathcal{H} \bar{L}_{ji} - \mathcal{K} \bar{M}_{[ij]} - \mathcal{K} (\bar{L}_{i1} - \bar{N}_{i0}) \delta_{2j} - \left(\mathcal{H} \delta_{ij} + \frac{1}{2} \mathcal{K}^2 \delta_{2i} \delta_{2j} \right) \frac{D\Omega}{\Omega}$$

$$- (\mathcal{K} (2\bar{L}_{(1k)} - \mathcal{H}\bar{L}_{k0} - \mathcal{K}\bar{\Xi}_k) + \delta_k \mathcal{K}) \delta_{2[i}\delta_{j]k} - \frac{1}{2}\mathcal{K}\frac{\delta_2\Omega}{\Omega}\delta_{ij} \quad (35)$$

with $\bar{\Xi}_i = \bar{L}_{2i} + \bar{M}_{i0}$ and $\bar{\Theta} = \bar{L}_{21} - \bar{N}_{20}$. It follows from (26) that the null vector \mathbf{k} is geodetic (and affinely parametrized) in the full spacetime g_{ab} if and only if it is geodetic (and affinely parametrized) in the background spacetime \bar{g}_{ab} . Subsequently, a geodetic \mathbf{k} has the same optical properties, i.e. the expansion, shear, and twist, in the both spacetimes since the optical matrices L_{ij} and \bar{L}_{ij} are equal.

2.2. Relation of the vector fields \mathbf{k} and \mathbf{m}

Inspired by the observation that the congruences \mathbf{k} and $\hat{\mathbf{m}} = \zeta\mathbf{m}$ of the CCLP black hole (discussed in section 5) obey (128), let us study the consequences of the covariant condition

$$(\zeta m_a)_{;b}k^b = (\zeta m_b)_{;a}k^b, \quad k_{a;b}m^b = k_{b;a}m^b, \quad (36)$$

restricting the geometry of the vectors \mathbf{k} and \mathbf{m} . The contractions of the first equation with the vectors \mathbf{n} , \mathbf{m} , $\mathbf{m}^{(\bar{i})}$ and the second equation with \mathbf{k} , \mathbf{n} , $\mathbf{m}^{(\bar{i})}$ give

$$L_{21} - N_{20} = 0, \quad L_{22} = -\frac{D\zeta}{\zeta}, \quad L_{2\bar{i}} + M_{\bar{i}0} = 0 \quad (37)$$

and

$$L_{20} = 0, \quad L_{[12]} = 0, \quad L_{[2\bar{i}]} = 0, \quad (38)$$

respectively. The Lie derivative of a one-form ω along a vector field X under the condition $X^a\omega_a = 0$ reads

$$(\mathcal{L}_X\omega)_a = \omega_{a;b}X^b + X^b_{;a}\omega_b = (\omega_{a;b} - \omega_{b;a})X^b \quad (39)$$

and therefore the relations (36) can be equivalently expressed as

$$\mathcal{L}_{\mathbf{k}}m_a = -\frac{D\zeta}{\zeta}m_a, \quad \mathcal{L}_{\mathbf{m}}k_a = 0. \quad (40)$$

It turns out that the additional covariant condition (36) restricts the geometry of the vectors \mathbf{k} and \mathbf{m} in the xKS metric (2) so that the implication of proposition 1 becomes an equivalence. Thus, due to (37) and (38), all the terms apart from the first two in (21) vanish

$$R_{00} = \left(2\mathcal{H} - \frac{1}{2}\mathcal{K}^2\right) L_{\bar{i}0}L_{\bar{i}0}, \quad (41)$$

which immediately implies that

Corollary 2 *In the special case when $\mathcal{K}^2 \neq 4\mathcal{H}$, $\mathcal{L}_{\mathbf{m}}k_a = 0$, and $\mathcal{L}_{\mathbf{k}}m_a \propto m_a$, the null vector field \mathbf{k} in the extended Kerr–Schild metric (2) is geodetic if and only if the boost weight 2 component of the Ricci tensor $R_{00} = R_{ab}k^ak^b$ vanishes.*

On the other hand, the boost weight zero component R_{00} vanishes regardless of whether \mathbf{k} is geodetic in case $\mathcal{K}^2 = 4\mathcal{H}$, $\mathcal{L}_{\mathbf{m}}k_a = 0$, and $\mathcal{L}_{\mathbf{k}}m_a \propto m_a$.

Notice that the Lie bracket of the vectors \mathbf{k} and \mathbf{m} expressed in terms of the Ricci rotation coefficients reads

$$[\mathbf{m}, \mathbf{k}]^a = L_{20} n^a + (L_{12} + N_{20})k^a + (L_{i2} - M_{i0})m_{(i)}^a. \quad (42)$$

If \mathbf{k} and \mathbf{m} satisfy the relation (36), then

$$[\mathbf{m}, \mathbf{k}]^a = 2L_{12} k^a + L_{22} m^a + 2L_{2\bar{i}} m_{(\bar{i})}^a \quad (43)$$

and therefore the vector fields \mathbf{k} and \mathbf{m} are surface-forming provided that $L_{2\bar{i}} = 0$. From (42) it follows that general \mathbf{k} and \mathbf{m} in Kundt xKS spacetimes are surface-forming for $M_{i0} = 0$ which is automatically satisfied if (36) holds. Effectively, $\zeta = 1$ in the relation (36) for Kundt spacetimes since (37) implies $D\zeta = 0$. One may also consider spacetimes with a recurrent null vector (RNV) field ℓ forming a subclass of Kundt spacetimes characterized by the vanishing of all components L_{ab} except for L_{11} , see Appendix B. In case that (36) is met, the vector fields $\mathbf{k} \equiv \ell$ and \mathbf{m} of such RNV xKS spacetimes commute, i.e. $[\mathbf{k}, \mathbf{m}] = 0$, and therefore the integral curves of \mathbf{k} and \mathbf{m} can be chosen as coordinates.

To conclude this section, let us point out the compatibility of the relation (36) and the optical constraint [9, 30]

$$L_{ik}L_{jk} = \frac{L_{lk}L_{lk}}{(n-2)\theta}S_{ij}, \quad (44)$$

which can be considered as a possible generalization of the shear-free part of the Goldberg–Sachs theorem to higher dimensions, see [18] for recent review. Although it has been shown that any five-dimensional algebraically special Einstein spacetime obeys the optical constraint [30], the situation in dimension $n > 5$ is not so clear. In fact, the optical constraint is satisfied at least for certain classes of spacetimes in arbitrary dimension such as, for instance, expanding Einstein GKS spacetimes [10], type N Einstein spacetimes and non-twisting type III Einstein spacetimes [13].

The optical constraint (44) implies that the optical matrix L_{ij} can be put to a block-diagonal form with 2×2 and 1×1 blocks using an appropriate spins. For a geodetic Weyl aligned null direction (WAND) ℓ , the r -dependence of L_{ij} satisfying the optical constraint (44) can be determined by integrating the Sachs equation [14]. Thus, one gets [9, 10]

$$L_{ij} = \text{diag} \left(\begin{bmatrix} s_1 & A_1 \\ -A_1 & s_1 \end{bmatrix}, \dots, \begin{bmatrix} s_p & A_p \\ -A_p & s_p \end{bmatrix}, \frac{1}{r}, \dots, \frac{1}{r}, 0, \dots, 0 \right), \quad (45)$$

where

$$s_\mu = \frac{r}{r^2 + a_\mu^2}, \quad A_\mu = \frac{a_\mu}{r^2 + a_\mu^2} \quad (46)$$

and a_μ are arbitrary functions not depending on r . Comparing (38), namely $L_{[2\bar{i}]} = 0$, with the optical matrix (45), it follows that the vector \mathbf{m} must not lie in any plane

spanned by two spacelike frame vectors corresponding to a 2×2 block with non-vanishing twist of the null geodetic congruence \mathbf{k} . Therefore, omitting the degenerate case $L_{22} = 0$, \mathbf{m} lies in a 1×1 block of the optical matrix, i.e. $L_{22} = r^{-1}$. From (37) we then obtain $\zeta = \alpha r^{-1}$, where α does not depend on the affine parameter r along null geodesics \mathbf{k} . It is shown in section 5 that for the CCLP black hole the vectors \mathbf{k} and \mathbf{m} satisfy the relation (36) and the optical constraint (44) also holds. In this case the function α corresponds to ν .

3. Geodetic Kerr–Schild vector field

From now on, we assume that \mathbf{k} is the tangent vector field of a null geodetic congruence and therefore $R_{00} = 0$ as follows from the proposition 1. Moreover, without loss of generality, the geodesics are considered to be affinely parametrized. In terms of the Ricci rotation coefficients, this means $L_{i0} = L_{10} = 0$ which considerably simplifies the following calculations. Note that in the case $\mathcal{K} = 0$, i.e. for GKS metrics, the assumption of Einstein spaces or spacetimes with aligned matter fields $T_{ab}k^b \propto k_a$ in the context of general relativity implies that \mathbf{k} is geodetic.

The frame components of the Ricci and Riemann tensors for xKS spacetimes (2) with a geodetic and affinely parametrized vector field \mathbf{k} are presented in Appendix A. All these components are much more complicated than in the case of GKS spacetimes, in particular the boost weight 1 components of the Ricci tensor R_{0i} (A.2) and the Riemann tensor R_{010i} (A.8), R_{0ijk} (A.9), respectively, no longer vanish identically. However, since the boost weight 2 components R_{00} (A.1) and R_{0i0j} (A.7) are zero, it follows that the same also holds for the Weyl tensor

$$C_{0i0j} = 0 \tag{47}$$

and therefore

Proposition 3 *Extended Kerr–Schild spacetimes (2) with a geodetic Kerr–Schild vector \mathbf{k} are of the Weyl type I with \mathbf{k} being the WAND.*

In general, xKS spacetimes with a geodetic \mathbf{k} are not necessarily of Weyl type II which confirms one of our motivations that these spacetimes may cover more general algebraic types than Einstein GKS spacetimes.

Employing the components R_{ij} of the Ricci tensor, one may show that non-expanding Einstein GKS spacetimes belong to the Kundt class (i.e. $\theta = 0$ implies $\sigma = \omega = 0$) and the optical matrix L_{ij} of expanding Einstein GKS spacetimes satisfies the optical constraint [9, 10]. In fact, these results hold for more general class of GKS spacetimes not necessarily Einstein (13), namely, it suffices to assume $R_{ij} = \frac{2\Lambda}{n-2}\delta_{ij}$ with R_{01} being arbitrary.

In the case of xKS spacetimes, the optical properties of the null congruence \mathbf{k} are not so restricted, neither if one assumes the relation (36) between the vectors \mathbf{k} and \mathbf{m} , and therefore non-expanding xKS spacetimes may have, in principle, non-vanishing shear and twist. More precisely, if a non-expanding geodetic null congruence ℓ of a

spacetime with $R_{ab}\ell^a\ell^b = 0$ is non-shearing, it is consequently non-twisting and vice versa [14]. For non-expanding xKS spacetimes, such a congruence is \mathbf{k} since $R_{00} = 0$ and thus xKS spacetimes with $\theta = 0$ are either Kundt or have both shear and twist non-vanishing.

In case the vectors \mathbf{k} and \mathbf{m} satisfy the relation (36) restricting the geometry of xKS spacetimes, the frame components of the Ricci and Riemann tensors further simplify. Then, due to (37) and (38), the boost weight 1 components of the Ricci tensor (A.2)

$$R_{02} = -\frac{1}{2}(L_{\bar{i}} + D)(DK + \mathcal{K}L_{22}) - \mathcal{K}\omega^2 - \mathcal{K}L_{2\bar{i}}L_{2\bar{i}} + \mathcal{K}L_{22}L_{\bar{i}\bar{i}}, \quad (48)$$

$$R_{0\bar{i}} = (DK + \mathcal{K}L_{22})L_{2\bar{i}} - \mathcal{K}L_{2\bar{j}}L_{\bar{j}\bar{i}} + \mathcal{K}L_{2\bar{i}}L_{\bar{j}\bar{j}} \quad (49)$$

and the Riemann tensor (A.8), (A.9)

$$R_{0102} = \frac{1}{2}D(DK + \mathcal{K}L_{22}), \quad (50)$$

$$R_{010\bar{i}} = -\frac{1}{2}(DK + \mathcal{K}L_{22})L_{2\bar{i}}, \quad (51)$$

$$R_{022\bar{i}} = -\frac{1}{2}(DK + \mathcal{K}L_{22})L_{2\bar{i}} + \mathcal{K}L_{2\bar{j}}A_{\bar{j}\bar{i}}, \quad (52)$$

$$R_{02\bar{i}\bar{j}} = -(DK + \mathcal{K}L_{22})A_{\bar{i}\bar{j}} + \mathcal{K}L_{[\bar{i}|\bar{k}}L_{\bar{k}|\bar{j}]}, \quad (53)$$

$$R_{0\bar{i}2\bar{j}} = -\frac{1}{2}(DK + \mathcal{K}L_{22})L_{\bar{i}\bar{j}} + \mathcal{K}L_{22}S_{\bar{i}\bar{j}} - \mathcal{K}L_{2\bar{i}}L_{2\bar{j}} + \mathcal{K}A_{\bar{i}\bar{k}}L_{\bar{k}\bar{j}}, \quad (54)$$

$$R_{0\bar{i}\bar{j}\bar{k}} = -2\mathcal{K}S_{\bar{i}[\bar{j}}L_{\bar{k}]\bar{2}}, \quad (55)$$

are given only in terms of the scalar function \mathcal{K} and the optical matrix L_{ij} . If all boost weight 1 components of the Riemann tensor (50)–(55) vanish, all boost weight 1 components of the Ricci and Weyl tensors vanish as well and, consequently, the spacetime is of Weyl type II. Obviously, this holds for $\mathcal{K} = 0$, when the xKS metric reduces to the GKS form studied in [10]. In the following, we assume $\mathcal{K} \neq 0$ and split $L_{\bar{i}\bar{j}}$ into its symmetric and anti-symmetric part, respectively. The independent conditions for type II then read

$$D\kappa = 0, \quad (56)$$

$$\kappa L_{\bar{i}} = 0, \quad (57)$$

$$A_{\bar{i}\bar{j}}L_{\bar{j}} = 0, \quad (58)$$

$$S_{\bar{i}[\bar{j}}L_{\bar{k}]} = 0, \quad (59)$$

$$\kappa A_{\bar{i}\bar{j}} = 2\mathcal{K}S_{[\bar{i}|\bar{k}}A_{\bar{k}|\bar{j}]}, \quad (60)$$

$$(2L_{22} - \kappa\mathcal{K}^{-1})S_{\bar{i}\bar{j}} = 2L_{\bar{i}}L_{\bar{j}} - 2A_{\bar{i}\bar{k}}A_{\bar{k}\bar{j}} - 2A_{(\bar{i}|\bar{k}}S_{\bar{k}|\bar{j})}, \quad (61)$$

where we denote $\kappa \equiv DK + \mathcal{K}L_{22}$ and $L_{\bar{i}} \equiv L_{2\bar{i}}$ for convenience. In order to solve these equations, several cases have to be investigated separately.

In the case $S_{\bar{i}\bar{j}} = 0$, we consider two subcases. If $\kappa = 0$, contracting (61) with $L_{\bar{j}}$ yields $L_{\bar{i}}L_{\bar{j}}L_{\bar{j}} = 0$ due to (58) and thus $L_{\bar{i}} = 0$. Then, the trace of (61) implies that the sum of squares of the elements of $A_{\bar{i}\bar{j}}$ vanish and therefore $A_{\bar{i}\bar{j}} = 0$. Otherwise, if $\kappa \neq 0$, it follows directly from (57), (60) that $L_{\bar{i}} = 0$, $A_{\bar{i}\bar{j}} = 0$ and it remains to solve (56).

In the case $S_{i\tilde{j}}$ is of rank 1, $S_{i\tilde{j}} = \text{diag}(s_{(3)}, 0, \dots, 0)$ and one gets from (59) that $L_\mu = 0$, where $\mu, \nu, \dots = 4, \dots, n-1$. If $\varkappa = 0$, it follows from (60) that $A_{3\mu} = 0$. Taking $\tilde{i} = \tilde{j} = \mu$ in (61) and summing over μ leads to $A_{\mu\nu} = 0$. Therefore, $A_{i\tilde{j}} = 0$ and L_3 is subject to $L_3^2 = L_{22}s_{(3)}$ as prescribed by (61). On the other hand, if $\varkappa \neq 0$, it immediately follows from (57) that $L_{\tilde{i}} = 0$ and from (60) that $A_{\mu\nu} = 0$. Putting $\tilde{i} = 3$, $\tilde{j} = \mu$ to (61) yields $A_{3\mu} = 0$ and thus $A_{i\tilde{j}} = 0$. Then (61) implies that $L_{22} = \mathcal{K}^{-1}D\mathcal{K}$, hence, $L_{22} \neq 0$ and it remains to satisfy $D^2\mathcal{K} = 0$ with $D\mathcal{K} \neq 0$.

In the case $m = \text{rank}(S_{i\tilde{j}}) \geq 2$, we can always set $S_{i\tilde{j}}$ to a diagonal form $S_{i\tilde{j}} = \text{diag}(s_{(3)}, \dots, s_{(m+2)}, 0, \dots, 0)$ by appropriate rotations of the spacelike frame vectors $\mathbf{m}^{(\tilde{i})}$ and therefore (59) leads to $L_{\tilde{i}} = 0$. In the following, it is convenient to employ indices $\alpha, \beta, \dots = 3, \dots, m+2$ and $\mu, \nu, \dots = m+3, \dots, n-1$ such that $s_{(\alpha)} \neq 0$, $s_{(\mu)} = 0$. Again, we consider two subcases with vanishing and non-vanishing \varkappa , respectively.

If $m \geq 2$, $\varkappa = 0$, from (60) for $\tilde{i} = \alpha$, $\tilde{j} = \mu$, it follows that $A_{\alpha\mu} = 0$ and then, from (61) for $\tilde{i} = \mu$, $\tilde{j} = \nu$, one obtains $A_{\mu\nu} = 0$. If, moreover, $L_{22} = 0$, (61) implies for $\tilde{i} = \tilde{j} = \alpha$ that $A_{\alpha\beta} = 0$. Therefore, $A_{i\tilde{j}}$ vanishes, $S_{i\tilde{j}}$ is an arbitrary diagonal matrix of rank m and $D\mathcal{K} = 0$. If, otherwise, $L_{22} \neq 0$, necessarily $L_{22}s_{(\alpha)} \neq 0$ and (61) for $\tilde{i} = \tilde{j} = \alpha$ yields $L_{22}s_{(\alpha)} = \sum_{\beta} A_{\alpha\beta}^2$, which means that $L_{22}s_{(\alpha)} > 0$ and for any given α there exists at least one β such that $A_{\alpha\beta} \neq 0$. However, (60) with $\tilde{i} = \alpha$, $\tilde{j} = \beta$ for non-vanishing $A_{\alpha\beta}$ implies $s_{(\alpha)} = -s_{(\beta)}$ and hence $L_{22}s_{(\beta)} < 0$, which is a contradiction and the case $\varkappa = 0$, $L_{22} \neq 0$ is thus excluded.

If $m \geq 2$, $\varkappa \neq 0$, from (60) for $\tilde{i} = \mu$, $\tilde{j} = \nu$, we get $A_{\mu\nu} = 0$ and then (61) for $\tilde{i} = \mu$, $\tilde{j} = \nu$ gives $A_{\alpha\mu} = 0$. Now, if $L_{22} = \frac{1}{2}\varkappa\mathcal{K}^{-1}$, putting $\tilde{i} = \tilde{j} = \alpha$ to (61) leads to $A_{\alpha\beta} = 0$. Therefore, $A_{i\tilde{j}}$ vanish, $s_{(\alpha)}$ are arbitrary and it remains to satisfy $D^2\mathcal{K} = 0$. On the other hand, if $L_{22} \neq \frac{1}{2}\varkappa\mathcal{K}^{-1}$, the parts of (60) and (61) involving $A_{\alpha\beta}$ can be written as

$$(\varkappa\mathcal{K}^{-1} - s_{(\alpha)} - s_{(\beta)})A_{\alpha\beta} = 0, \quad (62)$$

$$(L_{22} - \frac{1}{2}\varkappa\mathcal{K}^{-1})s_{(\alpha)} = \sum_{\beta} A_{\alpha\beta}A_{\alpha\beta}, \quad (63)$$

$$(s_{(\beta)} - s_{(\alpha)})A_{\alpha\beta} = 2 \sum_{\gamma} A_{\alpha\gamma}A_{\beta\gamma}, \quad \alpha \neq \beta, \quad (64)$$

where we do not use the summation convention over the repeated indices, instead the summation symbol is explicitly indicated for clarity. Any row of the submatrix $A_{\alpha\beta}$ contains at least one non-vanishing element since the left-hand side of (63) is non-vanishing. If we multiply (64) by $A_{\alpha\beta}$ and sum over β , the right hand-side vanishes due to the anti-symmetry of $A_{\beta\gamma}$ and we thus obtain

$$\sum_{\beta} (s_{(\beta)} - s_{(\alpha)})A_{\alpha\beta}A_{\alpha\beta} = 0. \quad (65)$$

Now, if a given row α contains just one non-vanishing element $A_{\alpha\beta}$, then $s_{(\alpha)} = s_{(\beta)} = \frac{1}{2}\varkappa\mathcal{K}^{-1}$ as follows from (65) and (62). If there are more $A_{\alpha\beta_l} \neq 0$ with a fixed α , then (62) implies that all the corresponding $s_{(\beta_l)}$ are equal and $s_{(\alpha)} = \varkappa\mathcal{K}^{-1} - s_{(\beta_l)}$. From

(65), one obtains that such $s_{(\beta_l)}$ are equal also to $s_{(\alpha)}$ and thus $s_{(\alpha)} = \frac{1}{2}\varkappa\mathcal{K}^{-1}$. Since this reasoning holds for any row of $A_{\alpha\beta}$, all $s_{(\alpha)} = \frac{1}{2}\varkappa\mathcal{K}^{-1}$. Therefore, (64) reduces to $\sum_{\gamma} A_{\alpha\gamma}A_{\beta\gamma} = 0$ pointing out that the row vectors are orthogonal, which along with (63) finally leads to $A_{\alpha\beta} = (\frac{1}{2}\varkappa\mathcal{K}^{-1}(L_{22} - \frac{1}{2}\varkappa\mathcal{K}^{-1}))^{\frac{1}{2}} O_{\alpha\beta}^A$, where $O_{\alpha\beta}^A$ is an arbitrary anti-symmetric orthogonal matrix. Since the determinant of any orthogonal matrix is 1 or -1 and the determinant of any regular $m \times m$ anti-symmetric matrix is positive for even m and vanishes for odd m , the dimension of the submatrix $A_{\alpha\beta}$ corresponding to the rank of S_{ij} has to be even. Note that $S_{\alpha\beta}$ is a multiple of the identity matrix and thus commute with $A_{\alpha\beta}$, as a consequence, one may simultaneously retain $S_{\alpha\beta}$ in the diagonal form $S_{\alpha\beta} = \frac{1}{2}\varkappa\mathcal{K}^{-1}\delta_{\alpha\beta}$ and put $A_{\alpha\beta}$ to a block diagonal form consisting of 2×2 blocks

$$A_{\alpha\beta} = \frac{\sqrt{\varkappa\mathcal{K}^{-1}(2L_{22} - \varkappa\mathcal{K}^{-1})}}{2} \text{diag} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \quad (66)$$

by appropriate rotations of the spacelike frame vectors $\mathbf{m}^{(i)}$.

For all the given forms of the optical matrix solving (57)–(61), one can integrate the Sachs equation [14] following from the Ricci identities which for xKS spacetimes with a geodesic \mathbf{k} reads

$$DL_{ij} = -L_{kj}^k M_{i0}^k - L_{ik}^k M_{j0}^k - L_{ik}L_{kj}. \quad (67)$$

Therefore, we are able to determine the r -dependence of these optical matrices and consequently of the corresponding functions \mathcal{K} as follows

$$L_{ij}^{(1)} = 0, \quad \mathcal{K}^{(1)} = c_1 r + c_2, \quad (68)$$

$$L_{ij}^{(2)} = \text{diag} \left(\frac{1}{r}, 0, \dots, 0 \right), \quad \mathcal{K}^{(2)} = c_1 r + \frac{c_2}{r}, \quad (69)$$

$$L_{ij}^{(3)} = \frac{1}{1 + c_1^2 r^2} \text{diag} \left(\begin{bmatrix} \frac{1}{r} & c_1 \\ c_1 & c_1^2 r \end{bmatrix}, 0, \dots, 0 \right), \quad \mathcal{K}^{(3)} = \frac{\sqrt{1 + c_1^2 r^2}}{c_2 r}, \quad c_1 \neq 0, \quad (70)$$

$$L_{ij}^{(4)} = \text{diag} \left(0, \frac{1}{r}, \frac{1}{r + c_2}, \dots, \frac{1}{r + c_p}, 0, \dots, 0 \right), \quad \text{rank}(L_{ij}^{(4)}) \geq 1, \quad \mathcal{K}^{(4)} = c_1, \quad (71)$$

$$L_{ij}^{(5)} = \text{diag} \left(\frac{1}{r}, \frac{1}{r + c_2}, \dots, \frac{1}{r + c_p}, 0, \dots, 0 \right), \quad \text{rank}(L_{ij}^{(5)}) \geq 2, \quad \mathcal{K}^{(5)} = c_1 r, \quad (72)$$

$$L_{ij}^{(6)} = \text{diag} \left(\frac{1}{r}, \mathcal{M}, \dots, \mathcal{M} \right), \quad \mathcal{K}^{(6)} = c_1 r + \frac{c_2}{r}, \quad (c_1 \neq 0) \wedge (c_2 \neq 0), \quad (73)$$

respectively, where the arbitrary functions c_μ independent of r are subject to $\mathcal{K} \neq 0$ and

$$\mathcal{M} = \begin{bmatrix} s & A \\ -A & s \end{bmatrix}, \quad s = \frac{r}{r^2 + \frac{c_2}{c_1}}, \quad A = \sqrt{\frac{c_2}{c_1}} \frac{1}{r^2 + \frac{c_2}{c_1}}. \quad (74)$$

We thus arrive at

Proposition 4 *In the case that $\mathcal{K} \neq 0$, $\mathcal{L}_{\mathbf{m}}k_a = 0$, and $\mathcal{L}_{\mathbf{k}}m_a \propto m_a$, extended Kerr–Schild spacetimes with a geodesic \mathbf{k} satisfying $R_{0i} = 0$ are algebraically special if and only if the optical matrix L_{ij} and the function \mathcal{K} can be put to any of the forms (68)–(73).*

Note that the case (68) belongs to the Kundt class. The optical matrix (70) can be set to the form $L_{ij} = \text{diag}(\frac{1}{r}, 0, \dots, 0)$ by an appropriate rotation in the plane spanned by \mathbf{m} and $\mathbf{m}^{(3)}$, therefore, the optical constraint holds for the cases (68)–(70) and (73). The optical matrices (71) and (72) satisfy the optical constraint only for $c_\mu = 0$ where $\mu = 2, \dots, p$ and if, moreover, the optical matrix (72) is regular such spacetimes belong to the Robinson–Trautmann class. In four dimensions, the cases (69)–(70) are excluded by the Goldberg–Sachs theorem. In five dimensions, the cases (69)–(72) with the corresponding optical matrices of rank 1 and 2, respectively, should be contained in the class of algebraically special non-twisting solutions found in [31].

4. Kundt extended Kerr–Schild spacetimes

The simplest subclass of xKS spacetimes with a geodesic Kerr–Schild vector \mathbf{k} is characterized by the vanishing of the optical matrix L_{ij} . In other words, the null geodesic congruence \mathbf{k} is non-expanding, non-shearing, and non-twisting, i.e. such xKS spacetimes belong to the Kundt class. The relevant components for the following analysis of the Ricci tensor (A.2)–(A.4) and the Riemann tensor (A.8)–(A.13) after substituting $L_{ij} = 0$ reduce to

$$R_{0i} = -\frac{1}{2}D^2\mathcal{K}\delta_{2i} - \frac{1}{2}\mathcal{K}DM_{i0} - M_{i0}DK + \frac{1}{2}\mathcal{K}M_{j0}\overset{i}{M}_{j0}, \quad (75)$$

$$R_{01} = -D^2\mathcal{H} + \frac{1}{2}\mathcal{K}D^2\mathcal{K} + \frac{1}{2}(DK)^2 - \frac{1}{2}\delta_2DK + \mathcal{K}^{-1}D(\mathcal{K}^2N_{20}) \\ - \frac{1}{2}M_{ii}DK - \frac{1}{2}\delta_i(\mathcal{K}M_{i0}) + \mathcal{K}M_{i0}N_{i0} - \frac{1}{2}\mathcal{K}M_{i0}\overset{i}{M}_{jj} + \frac{2\Lambda}{n-2}, \quad (76)$$

$$R_{22} = -\delta_2DK + \frac{1}{2}(DK)^2 + 2L_{21}DK + \mathcal{K}M_{k2}M_{k0} + \frac{2\Lambda}{n-2}, \quad (77)$$

$$R_{i2} = -\frac{1}{2}\delta_iDK - \frac{1}{2}M_{i2}DK - \frac{1}{2}\delta_2(\mathcal{K}M_{i0}) + L_{i1}DK + \frac{1}{2}\mathcal{K}M_{i0}DK \\ - \mathcal{K}\overset{k}{M}_{(\bar{i}2)}M_{k0} + 2\mathcal{K}L_{21}M_{i0}, \quad (78)$$

$$R_{i\bar{j}} = -DKM_{(\bar{i}\bar{j})} - \delta_{(\bar{i}}(\mathcal{K}M_{\bar{j})0}) + 2\mathcal{K}L_{(\bar{i}|1}M_{|\bar{j})0} - \mathcal{K}\overset{k}{M}_{(\bar{i}\bar{j})}M_{k0} \\ + \frac{1}{2}\mathcal{K}^2M_{i0}M_{j0} + \frac{2\Lambda}{n-2}\delta_{i\bar{j}}, \quad (79)$$

and

$$R_{010i} = \frac{1}{2}D^2\mathcal{K}\delta_{2i} + \frac{1}{2}\mathcal{K}DM_{i0} + M_{i0}DK - \frac{1}{2}\mathcal{K}\overset{i}{M}_{j0}M_{j0}, \quad (80)$$

$$R_{0ijk} = 0, \quad (81)$$

$$R_{0101} = D^2\mathcal{H} - \frac{1}{4}(DK)^2 - \mathcal{K}M_{i0}N_{i0} + D(\mathcal{K}L_{21} - \mathcal{K}N_{20}) - N_{20}DK \\ - \frac{1}{4}\mathcal{K}^2M_{i0}M_{i0} - \frac{2\Lambda}{(n-2)(n-1)}, \quad (82)$$

$$R_{01\bar{i}2} = \frac{1}{2}\delta_iDK - \frac{1}{2}\delta_2(\mathcal{K}M_{i0}) - \frac{1}{2}M_{i2}DK - \mathcal{K}M_{k0}\overset{k}{M}_{[\bar{i}2]}, \quad (83)$$

$$R_{01\bar{i}\bar{j}} = \delta_{[\bar{i}}(\mathcal{K}M_{\bar{j}]0}) - M_{[\bar{i}\bar{j}]}DK - \mathcal{K}M_{k0}\overset{k}{M}_{[\bar{i}\bar{j}]}, \quad (84)$$

$$R_{0212} = -\frac{1}{2}\delta_2 D\mathcal{K} + \frac{1}{4}(D\mathcal{K})^2 + L_{21}D\mathcal{K} + \frac{1}{2}\mathcal{K}M_{k0}M_{k2} + \frac{2\Lambda}{(n-2)(n-1)}, \quad (85)$$

$$R_{021\bar{i}} = -\frac{1}{2}\delta_{\bar{i}}D\mathcal{K} + \frac{1}{4}(2L_{\bar{i}1} + \mathcal{K}M_{\bar{i}0})D\mathcal{K} + \frac{1}{2}\mathcal{K}M_{\bar{i}0}L_{21} + \frac{1}{2}\mathcal{K}M_{k0}M_{k\bar{i}}, \quad (86)$$

$$R_{0\bar{i}12} = \frac{1}{4}(2L_{\bar{i}1} - 2M_{\bar{i}2} + \mathcal{K}M_{\bar{i}0})D\mathcal{K} - \frac{1}{2}\delta_2(\mathcal{K}M_{\bar{i}0}) + \frac{1}{2}\mathcal{K}\left(M_{\bar{i}0}L_{21} - M_{k0}M_{\bar{i}2}^k\right), \quad (87)$$

$$R_{0\bar{i}1\bar{j}} = -\frac{1}{2}M_{\bar{i}\bar{j}}D\mathcal{K} - \frac{1}{2}\delta_{\bar{j}}(\mathcal{K}M_{\bar{i}0}) + \mathcal{K}L_{(\bar{i}1}M_{\bar{j})0} - \frac{1}{2}\mathcal{K}M_{k0}M_{\bar{i}\bar{j}}^k + \frac{1}{4}\mathcal{K}^2M_{\bar{i}0}M_{\bar{j}0} + \frac{2\Lambda}{(n-2)(n-1)}\delta_{\bar{i}\bar{j}}, \quad (88)$$

$$R_{ijkl} = \frac{4\Lambda}{(n-1)(n-2)}\delta_{i[k}\delta_{l]j}, \quad (89)$$

respectively.

As follows from proposition 3, Kundt xKS metrics are of Weyl type I, i.e. all boost weight 2 components of the Weyl tensor vanish. The only non-trivial boost weight 1 components of the Riemann tensor (80) obey $R_{010i} = -R_{0i}$ and therefore the boost weight 1 components of the Weyl tensor are determined just by R_{0i}

$$C_{0ijk} = \frac{1}{n-2}(R_{0k}\delta_{ij} - R_{0j}\delta_{ik}), \quad C_{010i} = \frac{3-n}{n-2}R_{0i}. \quad (90)$$

Obviously, Kundt xKS spacetimes are of Weyl type II if and only if $R_{0i} = 0$. Note that the same statement holds even for general Kundt metrics not necessarily of the xKS form [14]. Using (75), the equations $R_{02} = 0$ and $R_{0\bar{i}} = 0$ for xKS spacetimes take the forms

$$D^2\mathcal{K} = \mathcal{K}M_{\bar{j}0}M_{\bar{j}0} \quad (91)$$

and

$$D(\mathcal{K}^2M_{\bar{i}0}) = \mathcal{K}^2M_{\bar{j}0}M_{\bar{j}0}^{\bar{i}}, \quad (92)$$

respectively. The trivial solution $\mathcal{K} = 0$ corresponds to the GKS limit where the components R_{0i} identically vanish [10]. Since the Ricci rotation coefficients \bar{M}_{ja}^i are antisymmetric in the indices i and j , we eliminate the term on the right-hand side of (92) by multiplying the equation with $2\mathcal{K}^2M_{\bar{i}0}$. The remaining term on the left-hand side can be rewritten so that we arrive at $D(\mathcal{K}^4M_{\bar{i}0}M_{\bar{i}0}) = 0$ implying

$$\mathcal{K}^4M_{\bar{i}0}M_{\bar{i}0} = (c^0)^2, \quad (93)$$

where the function c^0 does not depend on the affine parameter r along the null geodesics of the Kerr–Schild congruence \mathbf{k} . Substituting (93) to (91), we obtain $\mathcal{K}^3D^2\mathcal{K} = (c^0)^2$ determining the r -dependence of the function \mathcal{K} which has two distinct branches of

solutions

$$\mathcal{K} = d^0 \sqrt{(r + b^0)^2 + \frac{(c^0)^2}{(d^0)^4}} \quad \text{if } c^0 \neq 0, \quad (94)$$

$$\mathcal{K} = f^0 r + e^0 \quad \text{if } c^0 = 0, \quad (95)$$

where b^0 , d^0 , e^0 and f^0 are arbitrary functions not depending on r . Since we assume \mathcal{K} to be non-zero, c^0 vanishes if and only if all M_{i0} vanish as can be seen directly from (93). In the case $c^0 \neq 0$ when $M_{i0} \neq 0$ one may determine the r -dependence of M_{i0} . Without loss of generality, $\bar{M}_{\bar{j}0}$ in (92) can be transform away using spatial rotations of $\mathbf{m}^{(\bar{i})}$ with $\mathbf{m}^{(2)}$ fixed to obtain $D(\mathcal{K}^2 M_{i0}) = 0$. It is convenient to denote $\mathcal{K}^2 M_{i0} = (d^0)^2 \mu_{\bar{i}}$ and comparing this form with (94) we can express M_{i0}

$$M_{i0} = \frac{\mu_{\bar{i}}}{(r + b^0)^2 + \mu_{\bar{j}} \mu_{\bar{j}}}, \quad \mu_{\bar{i}} \mu_{\bar{i}} = \frac{(c^0)^2}{(d^0)^4}, \quad (96)$$

where $\mu_{\bar{i}}$ does not depend on the affine parameter r . Finally, we can conclude that

Proposition 5 *For Kundt extended Kerr–Schild spacetimes with $\mathcal{K} \neq 0$ and the vector field \mathbf{k} corresponding to the non-expanding, non-twisting, and non-shearing null geodesics the following statements are equivalent:*

- (i) *The boost weight 1 components $R_{0i} \equiv R_{ab} k^a m_{(i)}^b = 0$ of the Ricci tensor vanish,*
- (ii) *the spacetime is of Weyl type II or more special,*
- (iii) *the function \mathcal{K} takes the form (94) or (95) along with M_{i0} given by (96).*

Note that in the context of the Einstein gravity, the vanishing of R_{0i} corresponds to the vanishing of the components T_{0i} of the energy–momentum tensor.

We can proceed further and restrict type II Kundt xKS spacetimes to Weyl type III by satisfying the additional conditions $C_{ijkl} = C_{01ij} = 0$. In terms of the Riemann and Ricci tensors, these components of the Weyl tensor read $C_{01ij} = R_{01ij}$ and

$$C_{ijkl} = -\frac{2}{n-2} (\delta_{i[k} R_{l]j} + R_{i[k} \delta_{l]j}) + 2 \frac{R + 2\Lambda}{(n-1)(n-2)} \delta_{i[k} \delta_{l]j}, \quad (97)$$

where we have substituted R_{ijkl} for Kundt xKS spacetimes from (89). Recall that Λ represents a cosmological constant of the background spacetime \bar{g}_{ab} . Necessarily, $R_{ij} = R_{(i)(j)} \delta_{ij} = \text{diag}(R_{22}, R_{33}, \dots, R_{(n-1)(n-1)})$ for C_{ijkl} to vanish and consequently

$$R_{(i)(i)} + R_{(j)(j)} = \frac{R + 2\Lambda}{n-1} \quad \forall i, j : i \neq j. \quad (98)$$

For $n = 4$, it follows from (98) that R_{22} and R_{33} have to satisfy

$$\Lambda = -R_{01} + R_{22} + R_{33}. \quad (99)$$

In higher dimensions, (98) implies that $R_{(i)(i)} = R_{(j)(j)}$ for all i, j and

$$\Lambda \delta_{ij} = \frac{n}{2} R_{ij} - R_{01} \delta_{ij}. \quad (100)$$

Proposition 6 *Type II Kundt extended Kerr–Schild spacetimes with $\mathcal{K} \neq 0$ and \mathbf{k} corresponding to the non-expanding, non-twisting, and non-shearing null geodesics are of Weyl type III if and only if the following two statements hold:*

- (i) *The boost weight zero components of the Ricci tensor R_{ij} (77)–(79) are diagonal $R_{ij} = \text{diag}(R_{22}, R_{33}, \dots, R_{(n-1)(n-1)})$ and along with R_{01} (76) satisfy either (99) in four dimensions or (100) for $n > 4$,*
- (ii) *the components of the Riemann tensor R_{01ij} (83) and (84) vanish.*

Note that the statement (i) is met obviously for Einstein spacetimes $R_{ab} = \frac{2\Lambda}{n-2}g_{ab}$ in arbitrary dimension.

Let us mention an example. We start with type II Kundt xKS spacetimes and set $c^0 = f^0 = 0$, i.e. $M_{i0} = 0$ and $\mathcal{K} = e^0$. One can immediately see from (83), (84) that $R_{01ij} = 0$ and from (76)–(79) that $R_{01} = -D^2\mathcal{H} + \mathcal{K}DN_{20} + \frac{2\Lambda}{n-2}$ and $R_{ij} = \frac{2\Lambda}{n-2}\delta_{ij}$. If $D^2\mathcal{H} = \mathcal{K}DN_{20}$, both statements of proposition 6 are satisfied and therefore such spacetimes are of Weyl type III.

The relation between the vectors \mathbf{k} and \mathbf{m} (36) for Kundt xKS spacetimes read

$$L_{21} = N_{20}, \quad D\zeta = 0, \quad M_{i0} = 0, \quad L_{[12]} = 0. \quad (101)$$

Since M_{i0} vanishes, the relation is compatible only with \mathcal{K} of the form (95), i.e. a linear function of the affine parameter r , in the case of type II subclass. Let us consider such spacetimes and assume furthermore that $D\mathcal{K} = 0$ which ensures type II. The situation is the same as in the previous example, but now $DN_{20} = 0$ due to the relation (101) and the one of the Ricci identities [14] for Kundt spacetimes $DL_{21} = L_{i1}M_{i0} - C_{0102} + \frac{1}{n-2}R_{02}$. Therefore, these spacetimes are of type III if $D^2\mathcal{H} = 0$.

4.1. Explicit examples of Kundt xKS metrics

Here, we present explicit examples of Ricci-flat Kundt xKS spacetimes, namely the class of spacetimes with vanishing scalar invariants (VSI) [25, 26] which can be described by metrics of the form

$$ds^2 = 2du\,dr + 2H(u, r, x^k)du^2 + 2W_i(u, r, x^k)du\,dx^i + \delta_{ij}dx^i\,dx^j. \quad (102)$$

It is easy to see that the VSI metrics (102) belong to the class of xKS spacetimes (2) with the flat background metric $\bar{g}_{ab}dx^a\,dx^b = 2du\,dr + \delta_{ij}dx^i\,dx^j$ and the following identification

$$\mathcal{H} = -H, \quad \mathcal{K} = -\sqrt{W_i W_i}, \quad k_a dx^a = du, \quad m_a dx^a = \frac{W_i dx^i}{\sqrt{W_j W_j}}. \quad (103)$$

The operator $D = \partial_r$ then just corresponds to the derivative with respect to the affine parameter r of the non-expanding, non-shearing, and non-twisting null geodesics generated by the vector field \mathbf{k} .

It is known [25] that VSI spacetimes are of Weyl type III and admit only negative boost weight components of the Ricci tensor, i.e. $R_{00} = R_{0i} = R_{01} = R_{ij} = 0$. All VSI metrics with the Ricci tensor of types N and O have been given explicitly in [26].

The VSI class can be divided into two distinct subclasses with vanishing and non-vanishing quantity $L_{1i}L_{1i}$ denoted as $\epsilon = 0$ and $\epsilon = 1$, respectively, which differ by the canonical choices of the functions W_i . The subclass $\epsilon = 0$ corresponds to RNV spacetimes containing also pp -waves, see Appendix B. Assuming a particular algebraic type of the Weyl and Ricci tensors, the functions W_i and H are further constrained [26].

Note that for VSI spacetimes the statements (i) and (ii) of proposition 5 are clearly satisfied and therefore the function \mathcal{K} takes one of the forms (94) or (95) depending on $W_i(u, r, x^k)$. In the subclass $\epsilon = 0$, the functions W_i are given by [26]

$$W_2 = 0, \quad W_{\bar{i}} = W_{\bar{i}}^0(u, x^k), \quad (104)$$

where $W_{\bar{i}}^0$ satisfy some additional constraints and are independent on the coordinate r corresponding to an affine parameter along the geodetic integral curves of the null vector \mathbf{k} . It follows from (103) that \mathcal{K} is of the form (95) with $f^0 = 0$ and does not depend on r

$$\mathcal{K} = -\sqrt{W_{\bar{i}}^0 W_{\bar{i}}^0} = e^0. \quad (105)$$

Therefore M_{i0} vanish and if also $N_{20} = 0$, then the vector \mathbf{m} is parallelly transported along the null geodesics \mathbf{k} . In fact, non-vanishing N_{20} can be always transformed away, while M_{i0} remain unaffected, by a null rotation with \mathbf{k} fixed [14] setting $Dz_2 = -N_{20}$. Although this Lorentz transformation changes the vector \mathbf{m} as $\hat{\mathbf{m}} = \mathbf{m} - z_2 \mathbf{k}$, the xKS form (2) of the metric is preserved if we introduce a new function \mathcal{H} such that $\hat{\mathcal{H}} = \mathcal{H} + z_2 \mathcal{K}$.

The subclass $\epsilon = 1$ of VSI spacetimes is characterized by the canonical form of the functions $W_i(u, r, x^k)$ [26]

$$W_2 = -\frac{2}{x^2}r, \quad W_{\bar{i}} = W_{\bar{i}}^0. \quad (106)$$

In the special case where all $W_{\bar{i}}^0$ vanish, the function \mathcal{K} corresponds to (95) with $e^0 = 0$

$$\mathcal{K} = -\frac{2}{|x^2|}r = f^0 r. \quad (107)$$

As in the previous case, $M_{i0} = 0$ and N_{20} vanishes or can be set to zero, consequently, the vector \mathbf{m} is parallelly transported along \mathbf{k} . On the other hand, if at least one of $W_{\bar{i}}$ in (106) is non-zero, the function \mathcal{K} takes the form (94)

$$\mathcal{K} = -\sqrt{\frac{4}{(x^2)^2}r^2 + W_{\bar{i}}^0 W_{\bar{i}}^0}. \quad (108)$$

Comparing (94) with (108), it immediately follows that $b^0 = 0$, $d^0 = -\frac{2}{|x^2|}$, and $(c^0)^2 = \frac{(x^2)^2}{4}W_{\bar{i}}^0 W_{\bar{i}}^0$. Now, the vector \mathbf{m} is not parallelly transported along \mathbf{k} since $M_{i0}M_{i0} = \mathcal{K}^{-4}(c^0)^2 \neq 0$.

Table 1. Properties of higher dimensional Ricci-flat pp -waves. With regard to the particular algebraic type, such pp -waves belong to the various classes of spacetimes.

Weyl type	KS	xKS	VSI
N	✓	✓	✓
III=III(a)	×	✓	✓
II=II(abd)	×	CSI	×

4.2. Relation of higher dimensional pp -waves and the class of xKS spacetimes

In the previous section, we have shown that all VSI metrics admit the xKS form (2). The question is whether also all pp -waves belong to the class of xKS spacetimes. In our discussion, we restrict ourselves to Einstein spaces, i.e. vacuum solutions with a possible cosmological constant in the framework of general relativity.

Higher dimensional pp -waves are, in general, of Weyl type II and Einstein pp -waves are necessarily Ricci-flat as discussed in Appendix B. Furthermore, it is known that Kundt spacetimes of Weyl type III with the Ricci tensor of type III, including Weyl type III Ricci-flat pp -waves, belong to the VSI class [25]. Therefore, it remains to investigate Ricci-flat pp -waves of genuine type II.

Any pp -wave spacetime can be described by a metric [32]

$$ds^2 = 2du [dv + H(u, x^k) du + W_i(u, x^k) dx^i] + g_{ij}(u, x^k) dx^i dx^j, \quad (109)$$

which cannot be cast to the xKS form for any arbitrary transverse metric g_{ij} . However, the transverse Riemannian metric g_{ij} of vacuum pp -waves is Ricci-flat [33] and in the case of Kundt CSI metrics the transverse Riemannian space is locally homogeneous [34]. Since a Ricci-flat locally homogeneous Riemannian space is flat [35], we can conclude that Ricci-flat CSI pp -wave metrics can be written in the form (109) with flat transverse space, i.e. $g_{ij} = \delta_{ij}$, and thus belong to the class of xKS spacetimes with Minkowski background.

Recall also that type N Ricci-flat Kundt spacetimes and consequently vacuum type N pp -waves can be cast to the KS form (1) as was shown in [9]. All the above mentioned properties of higher dimensional vacuum pp -waves are summarized in table 1. On the other hand, the situation in four dimensions is much more simple since all vacuum pp -wave metrics are only of Weyl type N, belong to the VSI class, and take the KS form.

5. Examples of expanding extended Kerr–Schild spacetimes

In this section, we give an explicit example of expanding xKS metric, namely the CCLP solution [27]. For such a spacetime, we construct a null frame, show that the optical matrix obeys the optical constraint, and determine algebraic type of the Weyl tensor. The CCLP metric represents a charged rotating black hole in five-dimensional minimal gauged supergravity or equivalently in the Einstein–Maxwell–Chern–Simons theory with

a negative cosmological constant Λ and the Chern–Simons coefficient $\chi = 1$ described by the field equations

$$R_{ab} = \frac{2}{3}\Lambda g_{ab} + 2(F_{ac}F_b{}^c - \frac{1}{6}g_{ab}F_{cd}F^{cd}), \quad (110)$$

$$\nabla_b F^{ab} + \frac{\chi}{2\sqrt{3}\sqrt{-g}}\epsilon^{abcde}F_{bc}F_{de} = 0. \quad (111)$$

Note that the CCLP metric solves (110) and (111) also for a positive cosmological constant.

In fact, the xKS ansatz has been first proposed in [11] by showing that the CCLP black hole can be cast to the form

$$g_{ab} = \bar{g}_{ab} - 2\mathcal{H}k_a k_b - 2\hat{\mathcal{K}}k_{(a}\hat{m}_{b)}, \quad (112)$$

where we distinguish $\hat{\mathcal{K}}$, $\hat{\mathbf{m}}$ from \mathcal{K} , \mathbf{m} in our definition of the xKS ansatz (2) and (3) since the vector $\hat{\mathbf{m}}$ is not normalized to unity. In terms of spheroidal coordinates, the (Anti-)de Sitter background metric \bar{g}_{ab} and the vectors \mathbf{k} , $\hat{\mathbf{m}}$ are given by [11]

$$\begin{aligned} \bar{g}_{ab} dx^a dx^b = & - (1 - \lambda r^2) \frac{\Delta}{\Xi_a \Xi_b} dt^2 - 2dr \left(\frac{\Delta}{\Xi_a \Xi_b} dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) \\ & + \frac{\rho^2}{\Delta} d\theta^2 + \frac{(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi^2 + \frac{(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi^2, \end{aligned} \quad (113)$$

$$k_a dx^a = -\frac{\Delta}{\Xi_a \Xi_b} dt + \frac{a \sin^2 \theta}{\Xi_a} d\phi + \frac{b \cos^2 \theta}{\Xi_b} d\psi, \quad (114)$$

$$\hat{m}_a dx^a = \lambda ab \frac{\Delta}{\Xi_a \Xi_b} dt + \frac{b \sin^2 \theta}{\Xi_a} d\phi + \frac{a \cos^2 \theta}{\Xi_b} d\psi, \quad (115)$$

where $\lambda = \frac{\Lambda}{6}$, a and b are spins, Q corresponds to charge, r is the spheroidal radial coordinate, ϕ , ψ , θ are the angular coordinates with usual ranges $\phi \in \langle 0, 2\pi \rangle$, $\psi \in \langle 0, 2\pi \rangle$, $\theta \in \langle 0, \pi \rangle$, respectively, and

$$\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Xi_a = 1 + \lambda a^2, \quad (116)$$

$$\Xi_b = 1 + \lambda b^2, \quad \Delta = 1 + \lambda a^2 \cos^2 \theta + \lambda b^2 \sin^2 \theta. \quad (117)$$

The functions \mathcal{H} , $\hat{\mathcal{K}}$ and the one-form gauge potential proportional to the null Kerr–Schild vector \mathbf{k} then read

$$\mathcal{H} = -\frac{M}{\rho^2} + \frac{Q^2}{2\rho^4}, \quad \hat{\mathcal{K}} = -\frac{Q}{\rho^2}, \quad A = -\frac{\sqrt{3}Q}{2\rho^2}\mathbf{k}. \quad (118)$$

In order to put the CCLP metric into the xKS form with a unit vector \mathbf{m} , we rescale the vector $\hat{\mathbf{m}}$ and include its norm to the function \mathcal{K} so that

$$m_a dx^a = \frac{\lambda ab r}{\nu} \frac{\Delta}{\Xi_a \Xi_b} dt + \frac{br \sin^2 \theta}{\nu \Xi_a} d\phi + \frac{ar \cos^2 \theta}{\nu \Xi_b} d\psi, \quad (119)$$

$$\mathcal{K} = -\frac{Q\nu}{r\rho^2}, \quad (120)$$

where we define, for convenience, $\nu^2 \equiv \rho^2 - r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$.

We can simply choose \mathbf{k} and \mathbf{m} as the null and spacelike frame vectors ℓ and $\mathbf{m}^{(2)}$, respectively. The remaining frame vectors \mathbf{n} , $\mathbf{m}^{(3)}$ and $\mathbf{m}^{(4)}$ have to be determined by solving the constraints (14). One may find easier to construct the frame in the background spacetime since the metric \bar{g}_{ab} is not so complex as the full metric g_{ab} and subsequently employ (23). Such a frame can be expressed, for instance, as

$$k^a \partial_a = \partial_r, \quad (121)$$

$$m_{(2)}^a \partial_a = \frac{ab}{r\nu} \partial_t - \frac{Q\nu^2 + ab\rho^2}{r\rho^2\nu} \partial_r + \frac{b}{r\nu} \partial_\phi + \frac{a}{r\nu} \partial_\psi, \quad (122)$$

$$m_{(3)}^a \partial_a = \frac{\sqrt{\Delta} \sin \theta \cos \theta}{\rho^2} \left(\frac{a^2 - b^2}{\Delta} \partial_t + \frac{\Xi_a(r^2 + b^2) + \Xi_b(r^2 + a^2)}{\Pi} \partial_r \right) + \frac{\sqrt{\Delta}}{\rho^2} \left(\frac{a \cos \theta}{\sin \theta} \partial_\phi - \frac{b \sin \theta}{\cos \theta} \partial_\psi + r \partial_\theta \right), \quad (123)$$

$$m_{(4)}^a \partial_a = \frac{r\sqrt{\Delta}}{\nu\rho^2} \left(\frac{b^2 - a^2}{\Delta} \sin \theta \cos \theta (\partial_t - \Delta \partial_r) - \frac{a \cos \theta}{\sin \theta} \partial_\phi + \frac{b \sin \theta}{\cos \theta} \partial_\psi + \frac{\nu^2}{r} \partial_\theta \right), \quad (124)$$

$$n^a \partial_a = -\frac{1}{\rho^2} \left(r^2 + \frac{a^2 \Xi_a \cos^2 \theta - b^2 \Xi_b \sin^2 \theta}{\Pi} \right) \partial_t + \frac{1}{2\rho^2} \left(2M - \frac{Q^2}{\rho^2} + \lambda r^2 (r^2 + a^2 + b^2) - \frac{1}{\Pi^2} \left(\rho^2 - 4(2r^2 + a^2 + b^2) \sin^2 \theta \cos^2 \theta + \lambda \nu^2 (\nu^2 - 4(1 + \lambda) r^2) - \lambda a^2 b^2 (\Delta(1 + \Delta) - \lambda r^2 (1 - 2 \cos^2 \theta)^2) - \lambda (a^2 + b^2) ((4(a^2 + b^2 + \lambda a^2 b^2) + \lambda^2 r^2 \nu^2) \sin^2 \theta \cos^2 \theta - \nu^2 \Delta) - \lambda r^2 (a^2 \cos^2 \theta - b^2 \sin^2 \theta) (\Xi_a + \Xi_b) (1 - 2 \cos^2 \theta) + 4\lambda r^2 (a^2 \Xi_a \cos^6 \theta + b^2 \Xi_b \sin^6 \theta) \right) \right) \partial_r - \frac{a}{\rho^2} \left(1 - \lambda r^2 + \frac{\Delta \Xi_b}{\Pi} \right) \partial_\phi - \frac{b}{\rho^2} \left(1 - \lambda r^2 - \frac{\Delta \Xi_a}{\Pi} \right) \partial_\psi - \frac{r \Delta (\Xi_a + \Xi_b) \sin \theta \cos \theta}{\rho^2 \Pi} \partial_\theta, \quad (125)$$

where $\Pi = \Xi_a \cos^2 \theta - \Xi_b \sin^2 \theta$.

Having established the frame, we can straightforwardly calculate the Ricci rotation coefficients (17). It turns out that $L_{a0} = 0$, i.e. the Kerr–Schild congruence \mathbf{k} is geodetic and affinely parametrized, but only $\mathbf{m}^{(3)}$ and $\mathbf{m}^{(4)}$ are parallelly transported along \mathbf{k} since

$$N_{20} = -Q \frac{\nu}{\rho^4} \quad (126)$$

is non-vanishing. Interestingly, the optical matrix

$$L_{ij} = \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{r}{\rho^2} & \frac{\nu}{\rho^2} \\ 0 & -\frac{\nu}{\rho^2} & \frac{r}{\rho^2} \end{pmatrix} \quad (127)$$

takes the same block-diagonal form as in the case of uncharged five-dimensional Kerr–(A)dS black hole [10] and therefore L_{ij} also satisfies the optical constraint (44).

One may also show that the vectors \mathbf{k} and $\hat{\mathbf{m}}$ of the CCLP metric satisfy

$$(\hat{m}_{a;b} - \hat{m}_{b;a}) k^b = 0, \quad (k_{a;b} - k_{b;a}) \hat{m}^b = 0, \quad (128)$$

which holds not only in the full spacetime, but also in the background spacetime, i.e. regardless whether we use the covariant derivative compatible with the full metric g_{ab} or with the background metric \bar{g}_{ab} . The relation (128) immediately implies that \mathbf{k} and \mathbf{m} met (36) with $\zeta = |\hat{\mathbf{m}}| = \frac{\nu}{r}$. It can be seen directly from (127) that $L_{22} = \frac{1}{r}$, $L_{2\bar{i}} = 0$ and since from (37), (38) and (126) it follows that $L_{12} = -Q\frac{\nu}{\rho^4}$, the Lie bracket (43) then read

$$[\mathbf{k}, \mathbf{m}]^a = -2Q\frac{\nu}{\rho^4}k^a + \frac{1}{r}m^a \quad (129)$$

and therefore the vector fields \mathbf{k} and \mathbf{m} are surface-forming.

In agreement with proposition 3, the CCLP spacetime is of Weyl type I since the boost weight 2 components of the Weyl tensor expressed using the frame (121)–(125) completely vanish

$$C_{0i0j} = 0, \quad (130)$$

however, we show that it is not more special. First, let us assume that the CCLP spacetime is of type II, therefore, the Weyl tensor satisfies the corresponding Bel–Debever criterion [36] for this type

$$\ell_{[e}C_{a]b[cd}\ell_{f]}\ell^b = 0 \quad (131)$$

and we look for the multiple WAND $\ell = \ell^t\partial_t + \ell^r\partial_r + \ell^\theta\partial_\theta + \ell^\phi\partial_\phi + \ell^\psi\partial_\psi$. For simplicity, but without loss of generality, we consider only one spin a to be non-zero. The component $\ell_{[r}C_{\theta]c[\psi r}\ell_{\theta]}\ell^c$ vanishes if $\ell^t = \frac{a\sin^2\theta}{\Delta}\ell^\phi$. Moreover, $\ell_{[t}C_{\theta]c[\psi r}\ell_{\theta]}\ell^c = 0$ implies that $\ell^\phi = -\frac{a\cos\theta}{r\sin\theta}\ell^\theta$ and also that either $\ell^\theta \neq 0 \wedge \ell^\psi \neq 0$ and then ℓ^r can be expressed in terms of ℓ^θ and ℓ^ψ , or $\ell^\theta = \ell^\psi = 0$ and therefore $\ell = \ell^r\partial_r$. In the former case, ℓ cannot be a null vector since $\ell_a\ell^a = \rho^4\ell^{\theta^2} + \Delta r^4\cos^2\theta\ell^{\psi^2}$ is a sum of squares with positive coefficients. In the latter case, $\ell_{[t}C_{\theta]c[\psi t}\ell_{\theta]}\ell^c = 0$ implies that $\ell^r = 0$ and consequently $\ell = 0$. Thus, we can conclude that there is no multiple WAND ℓ satisfying the criterion (131) and the CCLP spacetime is of genuine type I.

Only if either $\nu = 0$, i.e. non-rotating limit, or in the uncharged case when the metric corresponds to the five-dimensional Kerr-(A)dS black hole, the metric reduces to the GKS form since \mathcal{K} (120) vanishes and the Weyl tensor is of type D.

Let us mention the results of [37] that stationary spacetimes with the metric remaining unchanged under reflection symmetry and with non-vanishing expansion are of Weyl types G, I_i, D or conformally flat. The CCLP metric obeys these conditions along with the reflection symmetry $t \rightarrow -t$, $\phi \rightarrow -\phi$, $\psi \rightarrow -\psi$ of the metric in Boyer–Lindquist-type coordinates given in [27]. Therefore, more specifically, the CCLP solution belongs to the subtype I_i of Weyl type I.

Finally, note also that we can adapt the frame (121)–(125) to be parallelly transported along the null geodesics \mathbf{k} using an appropriate null rotation with \mathbf{k} fixed. However, this operation changes $\mathbf{m}^{(2)}$ and thus breaks the identification $\mathbf{m} \equiv \mathbf{m}^{(2)}$. Such a Lorentz transformation which set N_{20} to zero is

$$\mathbf{k}' = \mathbf{k}, \quad \mathbf{m}' = \mathbf{m} - z_2\mathbf{k}, \quad \mathbf{m}'^{(\bar{i})} = \mathbf{m}^{(\bar{i})}, \quad \mathbf{n}' = \mathbf{n} + z_2\mathbf{m} - \frac{1}{2}z_2^2\mathbf{k}, \quad (132)$$

where

$$z_2 = \frac{Q}{2\nu^2} \left(\arctan \frac{r}{\nu} + \frac{\nu r}{\rho^2} \right). \quad (133)$$

Obviously, the transformation (132) preserves the xKS form of the CCLP metric with \mathbf{k}' and \mathbf{m}' being the corresponding null and spacelike unit vectors, respectively. The function \mathcal{K} remains unchanged and \mathcal{H} is now taken as

$$\mathcal{H}' = \mathcal{H} + z_2 \mathcal{K} = -\frac{1}{\rho^2} \left(M + \frac{Q^2}{2\nu r} \arctan \frac{r}{\nu} \right). \quad (134)$$

6. Conclusion

We have studied xKS spacetimes in any dimension $n \geq 4$ as a possible generalization of the well-known KS ansatz. Unlike the case $\mathcal{K} = 0$ corresponding to GKS spacetimes, the KS vector \mathbf{k} may not be geodesic neither for xKS spacetimes with aligned matter fields in the context of general relativity nor even for Einstein xKS spacetimes unless a special relation between the vectors \mathbf{k} and \mathbf{m} appearing in the xKS metric holds. It turns out that such a relation is compatible with the optical constraint and leads to further simplification of the Ricci and Riemann tensors of the xKS metric. Unlike the GKS case, xKS spacetimes with a geodesic \mathbf{k} are of Weyl type I and thus, in general, may not be algebraically special.

For the simplest subclass, namely the class of Kundt xKS spacetimes, we have been able to express the conditions for more special Weyl types determined by the specific r -dependence of the function \mathcal{K} and the form of the Ricci tensor. It turns out that VSI metrics including type III Ricci-flat pp -waves take the xKS form with flat background and thus a wide class of explicit examples of Kundt xKS spacetimes is known. Furthermore, it is shown that type II Ricci-flat pp -waves belong to the class of xKS spacetimes if they are CSI.

An example of expanding xKS spacetime, namely the CCLP solution representing a charged rotating black hole in five-dimensional minimal gauged supergravity, is also briefly discussed. We have established a null frame, expressed the optical matrix, and shown that the CCLP black hole is of Weyl type I_i . Interestingly, although the CCLP spacetime is not Einstein nor algebraic special, the corresponding optical matrix satisfies the optical constraint which, in fact, holds for any five-dimensional algebraically special Einstein spacetime [18].

We believe that the xKS form may lead to the discovery of new solutions of general relativity in higher dimensions in vacuum and also in the presence of matter fields aligned with the KS vector \mathbf{k} , such as an aligned Maxwell field. Using the xKS ansatz, one could also obtain new vacuum solutions in more general theories of gravity, for instance, in the Gauss–Bonnet theory or Lovelock gravities of higher order. We hope that the results of this paper will be useful for finding such new solutions in a subsequent work.

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Appendix A. Frame components of the Ricci and Riemann tensors

In this appendix section, we explicitly provide all components of the Ricci and Riemann tensors for xKS spacetimes (2) with respect to the frame (14). The frame vectors ℓ and $\mathbf{m}^{(2)}$ are identified with the vectors occurring in the xKS metric as $\mathbf{k} \equiv \ell$ and $\mathbf{m} \equiv \mathbf{m}^{(2)}$, respectively. The null Kerr–Schild vector \mathbf{k} is assumed to be geodetic and affinely parametrized.

Appendix A.1. Ricci tensor

$$R_{00} = 0, \quad (\text{A.1})$$

$$R_{0i} = -\frac{1}{2}(\text{D}^2\mathcal{K} + L_{jj}\text{D}\mathcal{K} + 2\omega^2\mathcal{K})\delta_{2i} - \frac{1}{2}(L_{jj} + \text{D})(\mathcal{K}\Xi_i) - \frac{1}{2}\Xi_i\text{D}\mathcal{K} - 2\mathcal{K}A_{2j}S_{ji} \\ - \mathcal{K}L_{2j}L_{ji} + \mathcal{K}L_{2i}L_{jj} + \frac{1}{2}\mathcal{K}(S_{ij} - 3A_{ij} + \overset{i}{M}_{j0})\Xi_j + \text{D}\mathcal{K}(L_{2i} + A_{2i}), \quad (\text{A.2})$$

$$R_{01} = -\text{D}^2\mathcal{H} - L_{ii}\text{D}\mathcal{H} - 2\mathcal{H}\omega^2 - \frac{1}{2}\delta_2\text{D}\mathcal{K} - A_{2i}\delta_i\mathcal{K} + \frac{1}{2}(L_{ii} + \text{D})(\mathcal{K}(\text{D}\mathcal{K} + \mathcal{K}L_{22})) \\ - \frac{1}{2}(2\mathcal{K}A_{2i} + \overset{i}{M}_{jj} + \delta_i)(\mathcal{K}\Xi_i) + 2N_{20}\text{D}\mathcal{K} + \mathcal{K}\text{D}N_{20} - \frac{1}{2}M_{ii}\text{D}\mathcal{K} \\ + \mathcal{K}(L_{i1}S_{2i} - L_{1i}A_{2i} - A_{ij}M_{ij} + L_{ii}N_{20} + M_{i0}N_{i0}) + \frac{2\Lambda}{n-2}, \quad (\text{A.3})$$

$$R_{ij} = -2S_{ij}\text{D}\mathcal{H} + 2\mathcal{H}L_{ik}L_{jk} - 2\mathcal{H}L_{kk}S_{ij} + \delta_{(i}(\mathcal{K}(2A_{2k} - \Xi_k))\delta_{k|j}) - S_{ij}\delta_2\mathcal{K} \\ + (\mathcal{K}S_{ij} - M_{(ij)})\text{D}\mathcal{K} + \mathcal{K}\left((2L_{[21]} + 2N_{20} - M_{kk})S_{ij} + 2M_{(i|0}L_{|j)1} + 2L_{(i|k}M_{|j)k} \right. \\ \left. - L_{kk}M_{(ij)} + (2A_{2k} - \Xi_k)\overset{k}{M}_{(ij)}\right) + \mathcal{K}^2\left((L_{22} + L_{kk})S_{ij} - S_{ik}S_{jk} - A_{ik}A_{jk} \right. \\ \left. + 2S_{2(i}A_{j)2} + \frac{1}{2}\Xi_i\Xi_j - L_{2(i}\Xi_{j)}\right) - \left[\delta_k\text{D}\mathcal{K} - 2L_{k1}\text{D}\mathcal{K} - 2L_{kl}\delta_l\mathcal{K} + \mathcal{K}(L_{k2} - \Xi_k)\text{D}\mathcal{K} \right. \\ \left. + L_{ll}\delta_k\mathcal{K} - 2\mathcal{K}\delta_lA_{kl} - 2\mathcal{K}\left(2L_{[1l]}S_{kl} - 2L_{l1}A_{kl} - L_{[1k]}L_{ll} + A_{kl}\overset{l}{M}_{mm} - A_{lm}\overset{k}{M}_{lm}\right) \right. \\ \left. - 2\mathcal{K}^2\left(2L_{[l}A_{k]2}L_{ll} - A_{kl}\Xi_l\right)\right]\delta_{2(i}\delta_{j)k} + \left[\frac{1}{2}(\text{D}\mathcal{K})^2 - \mathcal{K}^2\omega^2\right]\delta_{2i}\delta_{2j} + \frac{2\Lambda}{n-2}\delta_{ij}, \quad (\text{A.4})$$

$$R_{1i} = -\left(2L_{[1i]} + \delta_i\right)\text{D}\mathcal{H} + 2L_{ij}\delta_j\mathcal{H} - L_{jj}\delta_i\mathcal{H} + 2\mathcal{H}\left(\left(\overset{j}{M}_{kk} + \delta_j\right)A_{ij} + 3L_{[1j]}L_{ij} \right. \\ \left. + L_{(1j)}L_{ji} - L_{1i}L_{jj} + A_{jk}\overset{j}{M}_{ik}\right) + \frac{1}{2}\delta_i\delta_2\mathcal{K} - \delta_i\delta_2\mathcal{K} + \frac{1}{2}\text{D}\mathcal{K}\delta_i\mathcal{K} - \frac{1}{2}\left(S_{2i} - \Xi_i\right)\Delta\mathcal{K} \\ + \frac{1}{2}\left(N_{2i} + M_{i1} + \mathcal{K}L_{1i} - \mathcal{K}^2\left(L_{2i} - \frac{1}{2}\Xi_i\right)\right)\text{D}\mathcal{K} - \left(L_{(1i)} - \frac{1}{2}\mathcal{K}\left(L_{i2} - \Xi_i\right)\right)\delta_2\mathcal{K}$$

$$\begin{aligned}
 & -\left(L_{[12]} - N_{20} + \frac{1}{2}M_{jj} - \frac{1}{2}\mathcal{K}(L_{22} + L_{jj})\right)\delta_i\mathcal{K} + \left(M_{ij} - \frac{1}{2}\mathcal{K}L_{ij}\right)\delta_j\mathcal{K} \\
 & + \left(\mathcal{H}\mathcal{D}\mathcal{K} - \mathcal{K}\mathcal{D}\mathcal{H}\right)L_{2i} + \mathcal{K}\left[\delta_2L_{[1i]} + \left(\mathcal{K}(L_{2i} - 2\Xi_i) - 2\delta_i\right)L_{[12]} + \frac{1}{2}\Delta\Xi_i + \delta_jM_{[ij]}\right. \\
 & + 2\mathcal{H}\left(2L_{j[i}A_{j]2} + L_{j[i}\Xi_{j]}\right) - 2L_{[1|1}L_{|2]i} - \left(2L_{[12]} + M_{jj}\right)L_{(1i)} + \left(2M_{ij} + \dot{M}_{i2}^j\right)L_{[1j]} \\
 & + L_{(1j)}M_{ji} - \left(2L_{[1i]} + \mathcal{K}L_{2i} + \delta_i\right)\Theta - \left(2L_{ij} - L_{ji}\right)M_{j1} + 2L_{ij}N_{[j2]} - 2S_{ij}N_{(j2)} \\
 & + L_{jj}N_{2i} - \left(N_{[ij]} + \frac{1}{2}\dot{M}_{j1}^i\right)\Xi_j + \dot{M}_{ik}^jM_{[jk]} + \dot{M}_{kk}^jM_{[ij]} - 2\left(N_{ji} - \mathcal{K}M_{[ij]}\right)A_{2j} \\
 & + N_{j(i}\Xi_{j)} - \frac{1}{2}\mathcal{K}\left(L_{12} + \frac{1}{2}\mathcal{K}L_{22}\right)\left(2A_{2i} - \Xi_i\right) + \mathcal{K}L_{1[i}L_{j]j} - \mathcal{K}L_{j[i}L_{j]1} + \mathcal{K}L_{[i]j}M_{|j]2} \\
 & + \mathcal{K}M_{j[i}\Xi_{j]} + \frac{1}{2}\mathcal{K}L_{1i}\Xi_2 - \mathcal{K}^2S_{j[i}\Xi_{j]}\left] + \left[\frac{1}{2}\Delta\mathcal{D}\mathcal{K} - \frac{1}{2}\left(\mathcal{D}\mathcal{K} + \mathcal{K}S_{jj}\right)\delta_2\mathcal{K}\right. \right. \\
 & + \frac{1}{2}\left(2L_{[1j]} + \dot{M}_{kk}^j + \delta_j\right)\left(\delta_j\mathcal{K} + 2\mathcal{K}L_{[1j]}\right) + \frac{1}{2}\left(\frac{1}{2}\mathcal{K}\left(\mathcal{D}\mathcal{K} + \mathcal{K}L_{22}\right) + N_{jj}\right) \\
 & - \mathcal{K}\left(L_{12} - 2L_{21} + M_{jj}\right) - \left(2\mathcal{H} - \mathcal{K}^2\right)S_{jj}\mathcal{D}\mathcal{K} - \mathcal{K}\left(2\mathcal{H}A_{jk} + N_{jk} + \mathcal{K}M_{jk}\right)A_{jk} \\
 & \left. - \mathcal{K}^2\left(L_{[12]}S_{jj} + A_{2j}\left(L_{j1} + \frac{1}{2}\mathcal{K}\Xi_j\right)\right) + \frac{2\mathcal{K}\Lambda}{n-1}\right]\delta_{2i}, \tag{A.5}
 \end{aligned}$$

$$\begin{aligned}
 R_{11} = & \delta_i\delta_i\mathcal{H} + N_{ii}\mathcal{D}\mathcal{H} + \left(4L_{1i} - 2L_{i1} + \dot{M}_{jj}^i\right)\delta_i\mathcal{H} - S_{ii}\Delta\mathcal{H} + 2\mathcal{H}\left[\delta_iL_{1i} - \Delta S_{ii}\right. \\
 & + \left(4L_{[1i]} + \dot{M}_{jj}^i\right)L_{1i} - L_{ij}N_{ij} - 2\dot{M}_{j1}^iS_{ij}\left] - \frac{1}{2}\delta_i(\mathcal{K}\delta_i\mathcal{K}) - \Delta\delta_2\mathcal{K} + \left(\delta_2\mathcal{H} - N_{21}\right. \right. \\
 & + 4\mathcal{H}L_{[12]} + \mathcal{H}M_{ii} - \frac{1}{2}\mathcal{K}(N_{22} + N_{ii}) + \frac{1}{4}\mathcal{K}^2M_{22}\left.)\mathcal{D}\mathcal{K} + \mathcal{K}\left(\delta_i\mathcal{H} + 4\mathcal{H}L_{[1i]}\right)M_{i0}\right. \\
 & + \left(\mathcal{H} - \frac{1}{2}\mathcal{K}^2\right)\left(\delta_2\mathcal{D}\mathcal{K} + \delta_i(\mathcal{K}\Xi_i) - 2\delta_i(\mathcal{K}A_{2i}) + \mathcal{K}\dot{M}_{jj}^i\Xi_i + \mathcal{K}^2L_{2i}L_{2i} - \mathcal{K}^2L_{22}L_{ii}\right) \\
 & - \left(\delta_2\mathcal{K} + 2\mathcal{K}L_{[12]} + \mathcal{K}M_{ii}\right)\mathcal{D}\mathcal{H} + \mathcal{K}\Delta(\mathcal{K}S_{ii}) + \frac{1}{2}(\mathcal{K}\delta_2 - 2\Delta)(\mathcal{K}M_{ii}) - 2\Delta(\mathcal{K}L_{[12]}) \\
 & - \left(L_{11} - 2\mathcal{K}L_{12} + \frac{3}{2}\mathcal{K}L_{21} - \mathcal{K}N_{20} - \frac{1}{4}\mathcal{K}^2L_{22} + \frac{1}{2}\mathcal{K}^2S_{ii}\right)\delta_2\mathcal{K} - \left(2N_{2i} + 2\mathcal{K}L_{1i}\right. \\
 & - \frac{1}{2}\mathcal{K}L_{i1} + \frac{1}{2}\mathcal{K}\dot{M}_{jj}^i - \mathcal{K}M_{i2} - \mathcal{K}^2S_{2i} + \frac{3}{4}\mathcal{K}^2\Xi_i\left.)\delta_i\mathcal{K} - \mathcal{K}^2\left(\dot{M}_{jj}^i - \frac{1}{2}\mathcal{K}\Xi_i + \delta_i\right)L_{(1i)}\right. \\
 & - \mathcal{K}\left(3L_{1i} - 2L_{i1} + \dot{M}_{jj}^i + \mathcal{K}M_{i0} - \delta_i\right)N_{2i} + \mathcal{K}\left(\mathcal{H}\left(2L_{i1} - 2\dot{M}_{jj}^i + \mathcal{K}\Xi_i\right) - \mathcal{K}M_{i1}\right. \\
 & - \mathcal{K}^2\left(2L_{i1} + M_{i2} - \dot{M}_{jj}^i\right) + \mathcal{K}^3\left(A_{2i} - \Xi_i\right)\left.)A_{2i} - \mathcal{K}\left(2L_{11} - \mathcal{K}\left(2L_{12} - L_{21} + M_{ii}\right.\right.\right. \\
 & + 2N_{20} + \mathcal{K}\left(\frac{1}{2}L_{22} - S_{ii}\right) + \delta_2\left.)\right)L_{[12]} + \mathcal{K}\left(2N_{i1} - \mathcal{K}N_{i2} + 2\mathcal{K}^2L_{[1i]}\right)\left(S_{2i} - \frac{1}{2}\Xi_i\right) \\
 & - \mathcal{K}\left(L_{11} - \mathcal{K}N_{20}\right)M_{ii} - \mathcal{K}\left(2\dot{M}_{j1}^i - \mathcal{K}\dot{M}_{j2}^i\right)M_{(ij)} - \mathcal{K}N_{20}N_{ii} - \mathcal{K}M_{ij}N_{ij} \\
 & - \frac{1}{2}\mathcal{K}^2\left(2L_{1i} - L_{i1} - 2M_{i2} + \mathcal{K}\Xi_i\right)L_{1i} + \frac{1}{2}\mathcal{K}^2\left(L_{i1} - M_{i2}\right)L_{i1} + \frac{1}{2}\mathcal{K}^2M_{ij}M_{ji} \\
 & + \mathcal{K}^2\left(N_{ij} + 2\dot{M}_{j1}^i\right)S_{ij} + \frac{1}{2}\mathcal{K}^2N_{22}S_{ii} + \frac{1}{4}\mathcal{K}^3M_{i2}\Xi_i - \mathcal{K}^3A_{ij}M_{ij} - \frac{1}{2}\mathcal{K}^4A_{ij}A_{ij}
 \end{aligned}$$

$$+ \frac{(n-3)\Lambda}{(n-1)(n-2)} \mathcal{K}^2, \quad (\text{A.6})$$

where $\Xi_i = L_{2i} + M_{i0}$ and $\Theta = L_{21} - N_{20}$.

Appendix A.2. Riemann tensor

$$R_{0i0j} = 0, \quad (\text{A.7})$$

$$R_{010i} = \frac{1}{2} \mathcal{D}^2 \mathcal{K} \delta_{2i} + \frac{1}{2} \mathcal{D}(\mathcal{K} \Xi_i) + \frac{1}{2} M_{i0} \mathcal{D} \mathcal{K} - \frac{1}{2} \mathcal{K} \Xi_j \overset{i}{M}_{j0}, \quad (\text{A.8})$$

$$R_{0ijk} = \left(L_{i[j} \delta_{k]2} - A_{jk} \delta_{2i} \right) \mathcal{D} \mathcal{K} + \mathcal{K} \left(2L_{2[j} S_{k]i} + L_{i[j} \Xi_{k]} - A_{jk} \Xi_i - 2A_{il} L_{l[j} \delta_{k]2} \right. \\ \left. - 2S_{l[j} A_{k]l} \delta_{2i} \right), \quad (\text{A.9})$$

$$R_{0101} = \mathcal{D}^2 \mathcal{H} - \frac{1}{4} (\mathcal{D} \mathcal{K})^2 - \left(\frac{1}{2} \mathcal{K} L_{22} + N_{20} \right) \mathcal{D} \mathcal{K} + \mathcal{D}(\mathcal{K} \Theta) - \mathcal{K} \Xi_i N_{i0} - \frac{1}{4} \mathcal{K}^2 \Xi_i \Xi_i \\ - \frac{2\Lambda}{(n-2)(n-1)}, \quad (\text{A.10})$$

$$R_{01ij} = -2A_{ij} \mathcal{D} \mathcal{H} - 4\mathcal{H} S_{k[i} A_{j]k} + \delta_{[i} \mathcal{D} \mathcal{K} \delta_{j]2} + \left(\mathcal{K} L_{2[i} \delta_{j]2} - M_{[ij]} \right) \mathcal{D} \mathcal{K} - M_{[i0} \delta_{j]} \mathcal{K} \\ - (\delta_k \mathcal{K} + 2\mathcal{K} L_{[1k]}) L_{k[i} \delta_{j]2} + \mathcal{K} \left(L_{2[i} L_{1]j} + L_{2[i} L_{j]1} - 2A_{ij} \Theta - L_{k[i} M_{j]k} - \Xi_k \overset{k}{M}_{[ij]} \right. \\ \left. + L_{k[i} M_{j]k} + \delta_k \Xi_l \delta_{k[i} \delta_{j]l} \right) + \mathcal{K}^2 L_{2[i} \Xi_{j]}, \quad (\text{A.11})$$

$$R_{0i1j} = -L_{ij} \mathcal{D} \mathcal{H} + 2\mathcal{H} A_{ik} L_{kj} + \frac{1}{2} (\mathcal{K} S_{ij} - M_{ij}) \mathcal{D} \mathcal{K} - \frac{1}{2} L_{2j} \delta_i \mathcal{K} - \frac{1}{2} \delta_j (\mathcal{K} \Xi_i) \\ + \mathcal{K} \left(S_{ij} N_{20} - \Theta A_{ij} - L_{(1i} L_{2j} + \Xi_{(i} L_{j)1} + L_{kj} M_{[ik]} - \frac{1}{2} \Xi_k \overset{k}{M}_{ij} \right) + \frac{1}{2} \mathcal{K}^2 \left(L_{22} S_{ij} \right. \\ \left. - \Xi_i L_{2j} + \frac{1}{2} \Xi_i \Xi_j \right) - \frac{1}{2} \left[\delta_j \mathcal{D} \mathcal{K} + \mathcal{K} L_{2j} \mathcal{D} \mathcal{K} - L_{kj} \delta_k \mathcal{K} - 2\mathcal{K} L_{[1k]} L_{kj} \right] \delta_{2i} \\ + \left[\left(L_{l1} + \frac{1}{2} \mathcal{K} \Xi_l + \mathcal{K} A_{2l} \right) \mathcal{D} \mathcal{K} - 2\mathcal{K} L_{k1} A_{lk} - \mathcal{K}^2 A_{lk} \Xi_k \right] \delta_{2(i} \delta_{j)l} + \frac{1}{4} (\mathcal{D} \mathcal{K})^2 \delta_{2i} \delta_{2j} \\ + \frac{2\Lambda}{(n-2)(n-1)} \delta_{ij}, \quad (\text{A.12})$$

$$R_{ijkl} = 4\mathcal{H} \left(A_{ij} A_{kl} + A_{l[i} A_{j]k} + S_{l[i} S_{j]k} \right) + 2\mathcal{K} \left(A_{ij} M_{[kl]} + A_{kl} M_{[ij]} + A_{l[i} \overset{A}{M}_{j]k} \right. \\ \left. + M_{l[i} \overset{A}{A}_{j]k} + S_{l[i} \overset{S}{M}_{j]k} + M_{l[i} \overset{S}{S}_{j]k} \right) + 2\mathcal{K}^2 S_{k[i} S_{j]l} - 4\mathcal{K}^2 \delta_{2[i} A_{j]s} A_{s[k} \delta_{l]2} \\ + 2 \left[L_{[n|m} \delta_{p]} \mathcal{K} + A_{np} \delta_m \mathcal{K} + \mathcal{K} \left(\delta_m A_{np} - 2A_{s[n} \overset{s}{M}_{p]m} - L_{[1n]} L_{pm} + L_{[1p]} L_{nm} \right) \right. \\ \left. + 2\mathcal{K}^2 A_{2[n} S_{p]m} \right] \left(\delta_{2[i} \delta_{j]m} \delta_{nk} \delta_{pl} + \delta_{2[k} \delta_{l]m} \delta_{ni} \delta_{pj} \right) + \frac{4\Lambda}{(n-2)(n-1)} \delta_{i[k} \delta_{l]j}, \quad (\text{A.13})$$

$$R_{101i} = \delta_i \mathcal{D} \mathcal{H} + 2L_{[1i} \mathcal{D} \mathcal{H} - L_{ji} \delta_j \mathcal{H} + 2\mathcal{H} (L_{j1} S_{ij} - L_{1j} L_{ji}) - \mathcal{H} (A_{2i} \mathcal{D} \mathcal{K} - \mathcal{K} A_{ij} \Xi_j) \\ + \mathcal{K} L_{2i} \mathcal{D} \mathcal{H} - \left(\frac{1}{2} M_{i1} + N_{2i} - \frac{1}{4} \mathcal{K} M_{i2} + \frac{1}{4} \Xi_i \mathcal{K}^2 \right) \mathcal{D} \mathcal{K} + \delta_i (\mathcal{K} \Theta) - \frac{1}{2} \Delta (\mathcal{K} \Xi_i) \\ + L_{2i} \Delta \mathcal{K} - \frac{1}{4} \left(\mathcal{D} \mathcal{K} + 2L_{21} + \mathcal{K} L_{22} \right) \left(\delta_i \mathcal{K} + 2\mathcal{K} L_{(1i)} \right) + \mathcal{K} \left(2L_{[1i} + \mathcal{K} L_{2i} \right) \Theta$$

$$\begin{aligned}
 & +\mathcal{K}\left(L_{11}L_{2i}+L_{j1}M_{[ij]}+L_{ji}M_{j1}+L_{ji}N_{2j}\right)+\frac{1}{2}\mathcal{K}\left(\overset{i}{M}_{j1}+\mathcal{K}M_{[ij]}-2N_{ji}\right)\Xi_j \\
 & -\frac{1}{4}\mathcal{K}^2\left(2L_{21}+\mathcal{K}L_{22}\right)\Xi_i-\frac{1}{2}\left[\Delta DK+\frac{1}{2}\mathcal{K}(\mathrm{D}\mathcal{K})^2-\frac{1}{2}\mathrm{D}\mathcal{K}\delta_2\mathcal{K}-\frac{1}{2}\mathcal{K}\left(L_{12}-3L_{21}\right.\right. \\
 & \left.\left.-\mathcal{K}L_{22}\right)\mathrm{D}\mathcal{K}-\left(L_{j1}+\frac{1}{2}\mathcal{K}\Xi_j\right)\left(\delta_j\mathcal{K}+2\mathcal{K}L_{[1j]}\right)+\frac{4\mathcal{K}\Lambda}{(n-1)(n-2)}\right]\delta_{2i}, \quad (\text{A.14})
 \end{aligned}$$

$$\begin{aligned}
 R_{1ijk} = & 2L_{[j|i}\delta_{|k]}\mathcal{H}+2\delta_i(\mathcal{H}A_{jk})+4\mathcal{H}\left(L_{[1i]}A_{jk}+A_{i[j}L_{k]1}-L_{1[j}L_{k]i}-A_{l[j}\overset{l}{M}_{k]i}\right) \\
 & +2\mathcal{H}\mathcal{K}\left(S_{i[j}\Xi_{k]}+2A_{2[j}S_{k]i}\right)+M_{[j|i}\delta_{|k]}\mathcal{K}-\mathcal{K}S_{i[j}\delta_{k]}\mathcal{K}+M_{[jk]}\delta_i\mathcal{K}+\mathcal{K}\left(\delta_iM_{[jk]} \right. \\
 & +2L_{[1i]}M_{[jk]}-L_{1[j}M_{k]i}-L_{[j]1}M_{i|k]}+2A_{i[j}M_{k]1}-A_{jk}M_{i1}+2N_{2[j}L_{k]i}-A_{jk}N_{2i} \\
 & \left.-2M_{l[j}\overset{l}{M}_{k]i}-N_{[j|i}\Xi_{|k]}\right)-\mathcal{K}^2\left(\delta_{[j}A_{k]i}+\frac{1}{2}\delta_iA_{jk}-L_{1[j}L_{k]i}-L_{[j]1}L_{i|k]}-L_{[1i]}A_{jk} \right. \\
 & \left.+A_{il}\overset{l}{M}_{[jk]}-A_{jl}\overset{l}{M}_{[ik]}+A_{kl}\overset{l}{M}_{[ij]}-S_{i[j}M_{k]2}-M_{i[j}^{\mathrm{S}}\Xi_{k]}\right)-\mathcal{K}^3S_{i[j}\Xi_{k]}-\left[\delta_{[j}\delta_{k]}\mathcal{K} \right. \\
 & \left.-N_{[jk]}\mathrm{D}\mathcal{K}+\left(\mathcal{K}A_{2l}+\frac{1}{2}\mathcal{K}\Xi_l+L_{l1}\right)\delta_{l[j}\delta_{k]}\mathcal{K}-\overset{l}{M}_{[jk]}\delta_l\mathcal{K}+\mathcal{K}\left(\delta_jL_{[1k]}-\delta_kL_{[1j]} \right. \right. \\
 & \left. \left.-L_{1[j}L_{k]1}-2L_{[1l]}\overset{l}{M}_{[jk]}+2N_{l[j}A_{k]l}\right)+\mathcal{K}^2\left(2A_{2[j}L_{k]1}-\left(M_{lm}+\overset{m}{M}_{l2}\right)\delta_{l[j}A_{k]m} \right. \right. \\
 & \left. \left. +\left(\overset{l}{M}_{[jk]}+\delta_{l[j}\delta_{k]}\right)A_{2l}-\left(L_{[12]}+\frac{1}{2}\delta_2\right)A_{jk}+L_{[1l]}\Xi_{[j}\delta_{k]l}\right)+\mathcal{K}^3\left(A_{2[j}A_{k]2}+A_{2[j}\Xi_{k]} \right. \right. \\
 & \left. \left. -S_{2[j}S_{k]2}\right)\right]\delta_{2i}+\left[\delta_l\delta_i\mathcal{K}+\left((\mathcal{K}^2-2\mathcal{H})S_{il}+N_{il}-\mathcal{K}M_{(il)}\right)\mathrm{D}\mathcal{K}-\mathcal{K}S_{il}\delta_2\mathcal{K} \right. \\
 & +2\left(L_{[1i]}-\frac{1}{2}\mathcal{K}A_{2i}\right)\delta_l\mathcal{K}+\left(L_{1l}+\frac{1}{2}\mathcal{K}\Xi_l\right)\delta_i\mathcal{K}+\overset{m}{M}_{il}\delta_m\mathcal{K}+2\mathcal{K}\left(\delta_lL_{[1i]}+L_{[1i]}L_{1l} \right. \\
 & \left.+L_{[1m]}\overset{m}{M}_{il}+\left(2\mathcal{H}A_{ml}+N_{ml}\right)A_{im}\right)-2\mathcal{K}^2\left(\delta_{[i}A_{l]2}+\frac{1}{2}\delta_2A_{ik}+\left(L_{1l}-\frac{1}{2}\mathcal{K}\Xi_l\right)A_{2i} \right. \\
 & \left.+L_{[12]}L_{li}+\frac{1}{2}\left(M_{lm}+\overset{m}{M}_{l2}\right)A_{im}+A_{lm}\overset{m}{M}_{[2i]}+A_{2m}\overset{m}{M}_{[il]}+\left(A_{2l}-\frac{1}{2}\Xi_l\right)L_{[1i]}\right) \\
 & \left.-\frac{1}{2}\left(\left(\delta_l\mathcal{K}+2\mathcal{K}L_{[1l]}-2\mathcal{K}^2A_{2l}\right)\mathrm{D}\mathcal{K}-2\mathcal{K}\left(\delta_m\mathcal{K}+2\mathcal{K}L_{[1m]}\right)A_{lm}\right)\delta_{2i} \right. \\
 & \left. +\frac{4\mathcal{K}\Lambda}{(n-2)(n-1)}\delta_{il}\right]\delta_{2[j}\delta_{k]l}, \quad (\text{A.15})
 \end{aligned}$$

$$\begin{aligned}
 R_{1i1j} = & \delta_{(i}\delta_{j)}\mathcal{H}+N_{(ij)}\mathrm{D}\mathcal{H}-S_{ij}\Delta\mathcal{H}+4L_{1(i}\delta_{j)}\mathcal{H}-2L_{(i|1}\delta_{|j)}\mathcal{H}+\overset{k}{M}_{(ij)}\delta_k\mathcal{H} \\
 & -\mathcal{H}\left[2\Delta S_{ij}-2\delta_{(i|}L_{1|j)}-4L_{1i}L_{1j}+4L_{1(i}L_{j)1}-2L_{1k}\overset{k}{M}_{(ij)}-N_{k(i}A_{j)k}+2N_{k(i}S_{j)k} \right. \\
 & \left.+4S_{k(i}\overset{k}{M}_{j)1}-2\mathcal{H}A_{k(i}S_{j)k}\right]-\mathcal{K}\left(3A_{2k}+S_{2k}\right)\delta_{k(i}\delta_{j)}\mathcal{H}-\frac{1}{2}\mathcal{K}\delta_{(i}\delta_{j)}\mathcal{K}-\frac{1}{4}\delta_i\mathcal{K}\delta_j\mathcal{K} \\
 & -\frac{1}{2}\mathcal{K}N_{(ij)}\mathrm{D}\mathcal{K}+\mathcal{K}\Delta(\mathcal{K}S_{ij})-\Delta(\mathcal{K}M_{(ij)})+(\mathcal{H}\mathrm{D}\mathcal{K}-\mathcal{K}\mathrm{D}\mathcal{H})M_{(ij)}-\left[M_{k1}+2N_{2k} \right. \\
 & \left.-\mathcal{H}L_{k2}-\mathcal{K}\left(M_{k2}-\frac{3}{2}L_{1k}\right)-\frac{1}{2}\mathcal{K}^2\left(3A_{2k}+S_{2k}-\frac{5}{2}\Xi_k\right)\right]\delta_{k(i}\delta_{j)}\mathcal{K}+\Xi_{(i}\delta_{j)}(\mathcal{H}\mathcal{K}) \\
 & -\frac{1}{2}\mathcal{K}^2S_{ij}\delta_2\mathcal{K}-\frac{1}{2}\mathcal{K}\overset{\bar{k}}{M}_{(ij)}\delta_{\bar{k}}\mathcal{K}-\mathcal{K}\left[\delta_{(i|}N_{2|j)}+L_{11}M_{(ij)}+N_{20}N_{(ij)}+M_{k(i}N_{k|j)} \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\Xi_k - 2L_{2k} \right) \delta_{k(i} N_{j)1} + L_{[1k]} \left(2N_{2l} + M_{l1} \right) \delta_{k(i} \delta_{j)l} + \left(L_{1(i} \delta_{j)k} + \overset{k}{M}_{(ij)} \right) N_{2k} \\
 & + 2M_{k(i}^S \overset{k}{M}_{j)1} \Big] - \mathcal{K}^2 \left[\left(\delta_k L_{(1l)} - \left(M_{k2} - \frac{3}{2} L_{(1k)} \right) L_{[1l]} \right) \delta_{k(i} \delta_{j)l} + \left(\Theta - \frac{1}{2} \delta_2 \right) M_{(ij)} \right. \\
 & - A_{2(i} \delta_{j)k} \left(M_{k1} - 2N_{2k} \right) - 2L_{2(i} \delta_{j)k} N_{[2k]} + L_{(1\tilde{k})} \overset{\tilde{k}}{M}_{(ij)} - \frac{1}{2} M_{ik}^S M_{kj} + \frac{1}{2} M_{ik}^A M_{jk} \\
 & - \left(N_{kl} + 2\overset{k}{M}_{l1} \right) S_{k(i} \delta_{j)l} - \overset{k}{M}_{(i|2} M_{j)k}^S - \frac{1}{2} N_{22} S_{ij} + \left(N_{2k} - \frac{1}{2} N_{k2} \right) \Xi_{(i} \delta_{j)k} \Big] \\
 & - \mathcal{K}^3 \left[\delta_{(i} A_{j)2} + \frac{1}{2} \delta_{(i} \Xi_{j)} + \left(L_{[12]} + \frac{1}{2} M_{22} \right) S_{ij} - S_{k(i} M_{j)k}^A + 3A_{2(i} M_{j)2}^A - \frac{1}{2} M_{k(i} A_{j)k} \right. \\
 & - \frac{1}{2} \left(L_{2k} - \frac{3}{2} \Xi_k \right) L_{[1l]} \delta_{k(i} \delta_{j)l} - \left(A_{2k} - \frac{1}{2} \Xi_k \right) \overset{k}{M}_{(ij)} \Big] - \mathcal{K}^4 \left[A_{2(i} S_{j)2} + \frac{1}{2} A_{ik} A_{jk} \right. \\
 & - \frac{1}{2} S_{22} S_{ij} + \frac{1}{2} S_{2i} S_{2j} \Big] - \mathcal{HK} \left[\left(2L_{1k} + \delta_k \right) A_{2(i} \delta_{j)k} - \left(3A_{2k} - S_{2k} \right) L_{[1l]} \delta_{k(i} \delta_{j)l} \right. \\
 & - \left(2L_{[1k]} + \delta_k \right) \Xi_{(i} \delta_{j)k} - \left(\Xi_k - 2A_{2k} \right) \overset{k}{M}_{(ij)} - 2A_{k(i} M_{j)k}^S + 2S_{k(i} M_{j)k}^A - \mathcal{K} \left(3A_{2i} A_{2j} \right. \\
 & \left. \left. + S_{2i} S_{2j} - S_{22} S_{ij} - A_{2(i} \Xi_{j)} + 2S_{\tilde{k}(i} A_{j)\tilde{k}} \right) \right] - \left[\Delta \delta_k \mathcal{K} + \mathcal{D} \mathcal{H} \delta_k \mathcal{K} - \mathcal{D} \mathcal{K} \delta_k \mathcal{H} \right. \\
 & - 2\mathcal{K} A_{kl} \delta_l \mathcal{H} - \left(\mathcal{H} - \frac{1}{2} \mathcal{K}^2 \right) \delta_k \mathcal{D} \mathcal{K} + \frac{1}{4} \mathcal{K} \mathcal{D} \mathcal{K} \delta_k \mathcal{K} + 2\mathcal{K} \left(L_{[1k]} - \mathcal{K} A_{2k} \right) \mathcal{D} \mathcal{H} \\
 & - \frac{1}{4} \delta_k \delta_2 \mathcal{K}^2 + \left(N_{k1} - 4\mathcal{H} L_{[1k]} + \frac{1}{2} \mathcal{K} N_{2k} + 3\mathcal{H} \mathcal{K} A_{2k} + \frac{1}{2} \mathcal{K}^2 L_{[1k]} - \frac{1}{2} \mathcal{K}^2 M_{(2k)} \right) \mathcal{D} \mathcal{K} \\
 & + \left(\overset{l}{M}_{k1} + \left(\mathcal{H} - \frac{1}{2} \mathcal{K}^2 \right) L_{kl} + \frac{1}{2} \mathcal{K} M_{kl} + \frac{3}{2} \mathcal{K}^2 A_{kl} \right) \delta_l \mathcal{K} + \left(L_{11} + \mathcal{K} \Theta - \mathcal{K} L_{12} \right) \delta_k \mathcal{K} \\
 & - \mathcal{K} \left(L_{1k} - \frac{1}{2} L_{k1} - \frac{5}{2} \mathcal{K} A_{2k} + \frac{1}{4} \mathcal{K} \Xi_k \right) \delta_2 \mathcal{K} + 2 \left(L_{11} - \mathcal{K} N_{20} + \Delta \right) \left(\mathcal{K} L_{[1k]} - \mathcal{K}^2 A_{2k} \right) \\
 & - \mathcal{K}^2 \left(2L_{1k} - L_{k1} - 3\mathcal{K} A_{2k} + \frac{1}{2} \mathcal{K} \Xi_k + \delta_k \right) L_{[12]} - \mathcal{K}^2 \left(N_{lk} + 2\overset{l}{M}_{k1} - \mathcal{K} M_{[lk]} \right. \\
 & - \mathcal{K} \overset{l}{M}_{k2} - \mathcal{K} \delta_{kl} \delta_2 \Big) A_{2l} + 2\mathcal{K} \left(N_{l1} + \mathcal{K} N_{2l} - 2\mathcal{H} L_{l1} + \mathcal{K} \left(\mathcal{H} - \frac{1}{2} \mathcal{K}^2 \right) \left(A_{2l} - \Xi_l \right) \right. \\
 & \left. \left. + \mathcal{K}^2 L_{(1l)} \right) A_{kl} + \mathcal{K} \left(2\overset{l}{M}_{k1} + \mathcal{K} M_{kl} + 2\mathcal{H} L_{kl} - \mathcal{K}^2 S_{kl} \right) L_{[1l]} \right] \delta_{2(i} \delta_{j)k} \\
 & + \frac{1}{4} \left[\mathcal{K} \left(\delta_2 \mathcal{K} + 2\mathcal{K}^2 L_{[12]} \right) \mathcal{D} \mathcal{K} - \left(\delta_k \mathcal{K} + 2\mathcal{K} L_{[1k]} - \mathcal{K}^2 A_{2k} \right)^2 - \mathcal{K}^4 A_{2k} A_{2k} \right] \delta_{2i} \delta_{2j} \\
 & + \frac{\mathcal{K}^2 \Lambda}{(n-2)(n-1)} (\delta_{ij} - \delta_{2i} \delta_{2j}), \tag{A.16}
 \end{aligned}$$

where $M_{ij}^S = M_{(ij)}$, $M_{ij}^A = M_{[ij]}$, $\Xi_i = L_{2i} + M_{i0}$, and $\Theta = L_{21} - N_{20}$.

Appendix B. Kundt spacetimes

The Kundt class is defined geometrically as spacetimes admitting a congruence of non-expanding, non-shearing, and non-twisting null geodesics ℓ , i.e. $L_{i0} = L_{ij} = 0$. It is always possible to choose an affine parametrization so that $L_{10} = 0$. Furthermore,

employing the remaining freedom of the choice of the null frame, we can set $L_{1i} = L_{i1}$ and therefore the covariant derivative of ℓ read [38]

$$\ell_{a;b} = L_{11}\ell_a\ell_b + 2L_{1i}\ell_{(a}m_{b)}^{(i)}. \quad (\text{B.1})$$

In fact, one may also transform away all L_{1i} except one [2], but it is not necessary for the following discussion. Note also that if L_{1i} is non-zero, L_{11} can be always eliminated using null rotations with ℓ fixed.

Kundt spacetimes in arbitrary dimension have been recently studied from various points of view [33, 34, 38–40] and they prove to be important in higher order theories of gravity since the class of Einstein Kundt spacetimes contains universal metrics [41] and also some non-Einstein Kundt spacetimes are vacuum solutions of quadratic gravity [42]. Although the algebraic classification of the Kundt class was completely presented in terms of constraints on the metric functions in [40], for certain purposes the following procedure could be more convenient. Without imposing an explicit form of the metric, Kundt spacetimes can be classified according to the canonical form of $\ell_{a;b}$, which have a clear geometric interpretation, and for each subclass the Weyl type can be determined in terms of L_{11} , L_{1i} and the Ricci tensor.

The subclass $L_{1i} = 0$ represents spacetimes with a recurrent null vector (RNV) field. Obviously, the direction of such a recurrent vector ℓ remains invariant under parallel transport along any curve. The holonomy group of RNV spacetimes is $Sim(n-2)$ [43]. Furthermore, if L_{11} also vanishes, spacetimes belonging to this class admit a covariantly constant null vector (CCNV) ℓ and are referred to as *pp*-waves. In this case, ℓ is parallelly transported along any curve and the holonomy group specializes to Euclidean group $E(n-2)$ or a subgroup thereof.

Using (B.1) along with the Ricci identities and their appropriate contractions, we can directly obtain relevant components of the Ricci tensor

$$R_{00} = 0, \quad R_{0i} = DL_{1i} + L_{1k}^k M_{i0}, \quad (\text{B.2})$$

$$R_{01} = DL_{11} + \delta_i L_{1i} - 2L_{1i} N_{i0} + L_{1i}^i M_{jj} \quad (\text{B.3})$$

and the Riemann tensor

$$R_{0i0j} = R_{0ijk} = 0, \quad R_{010i} = -DL_{1i} - L_{1j}^j M_{i0}, \quad (\text{B.4})$$

$$R_{0i1j} = \delta_j L_{1i} + L_{1k}^k M_{ij} - L_{1j} L_{1i}, \quad (\text{B.5})$$

$$R_{01ij} = -2\delta_{[i} L_{1|j]} + 2L_{1k}^k M_{[ij]}, \quad (\text{B.6})$$

$$R_{0101} = -DL_{11} - L_{1i} L_{1i} + 2L_{1i} N_{i0}, \quad (\text{B.7})$$

$$R_{011i} = \delta_i L_{11} - 2L_{1j} N_{ji} - \Delta L_{1i} - L_{1j}^j M_{i1}, \quad (\text{B.8})$$

respectively. Subsequently, it allows us to express also the corresponding components of the Weyl tensor

$$C_{0i0j} = 0, \quad C_{0ijk} = \frac{2}{n-2} (DL_{1l} + L_{1m}^m M_{l0}) \delta_{i[j} \delta_{k]l}, \quad (\text{B.9})$$

$$C_{010i} = -\frac{n-3}{n-2}(DL_{1i} + L_{1j}\overset{j}{M}_{i0}), \quad (\text{B.10})$$

$$C_{0i1j} = -\frac{1}{n-2}(DL_{11} + \delta_k L_{1k} - 2L_{1k}N_{k0} + L_{1k}\overset{k}{M}_{ll})\delta_{ij} \\ + \delta_j L_{1i} + L_{1k}\overset{k}{M}_{ij} - L_{1j}L_{1i} - \frac{1}{n-2}R_{ij} + \frac{1}{(n-2)(n-1)}R\delta_{ij}, \quad (\text{B.11})$$

$$C_{01ij} = -2\delta_{[i}L_{1|j]} + 2L_{1k}\overset{k}{M}_{[ij]}, \quad (\text{B.12})$$

$$C_{0101} = -\frac{n-4}{n-2}(DL_{11} - 2L_{1i}N_{i0}) + \frac{2}{n-2}(\delta_i L_{1i} + L_{1i}\overset{i}{M}_{jj}) \\ - L_{1i}L_{1i} - \frac{1}{(n-2)(n-1)}R, \quad (\text{B.13})$$

$$C_{011i} = \delta_i L_{11} - 2L_{1j}N_{ji} - \Delta L_{1i} - L_{1j}\overset{j}{M}_{i1} - \frac{1}{n-2}R_{1i}. \quad (\text{B.14})$$

Now, for each subclass of Kundt spacetimes, we inspect these components of the Weyl tensor and compare them with the classification scheme based on null alignment and the spin-type refinement [15, 16], see also [18] for recent review.

It is known that Kundt spacetimes are in general of type I with $R_{00} = 0$ and of type II if and only if $R_{0i} = 0$ [14]. More precisely, type I Kundt spacetimes are of subtype I(b) since $C_{0ijk}C_{0ijk} = \frac{2}{n-3}C_{010i}C_{010i}$ as follows from (B.9) and (B.10).

For $L_{1i} = 0$, the components of the Ricci tensor (B.2) and (B.3) reduce to $R_{00} = R_{0i} = 0$, $R_{01} = DL_{11}$ which also implies that RNV spacetimes are of Weyl type II. The conditions for each possible subtype are discussed in more detail in table B1. It turns out that RNV spacetimes are in general of type II(d), while Einstein RNV spacetimes for which $DL_{11} = \frac{2\Lambda}{n-2}$ are of type II(bd). Ricci-flat RNV spacetimes are of type II(abd) with $DL_{11} = 0$ and conversely type II(a) Einstein RNV spacetimes are necessarily Ricci-flat. It was pointed out in [41] that for type III(a) Ricci-flat RNV spacetimes, L_{11} can be always transformed away by a boost (15) where λ subject to $L_{11} = -\lambda^{-1}\Delta\lambda$ since $DL_{11} = \delta_i L_{11} = 0$ and thus such spacetimes are pp -waves. Therefore, the entire class of type II(a) Einstein RNV spacetimes consists only of Ricci-flat CCNV spacetimes.

The conditions for types II(c) and III(b) are not given only in terms of L_{11} and the Ricci tensor. In order to distinguish these subtypes one has to necessarily know a particular form of the metric to express C_{ijkl} and C_{1ijk} , respectively. Type II RNV spacetimes are of subtype II(c) if and only if

$$C_{ijkl} = \frac{4}{(n-3)(n-4)}\left(DL_{11} - \frac{2n-5}{(n-1)(n-2)}R\right)\delta_{i[k}\delta_{l]j} \\ + \frac{4}{(n-2)(n-4)}(\delta_{i[k}R_{l]j} - \delta_{j[k}R_{l]i}). \quad (\text{B.15})$$

The necessary and sufficient condition for type III RNV spacetimes to be of subtype III(b) is

$$C_{1ijk}C_{1ijk} = \frac{2}{n-3}\left(\delta_i L_{11} - \frac{1}{n-2}R_{1i}\right)\left(\delta_i L_{11} - \frac{1}{n-2}R_{1i}\right). \quad (\text{B.16})$$

Table B1. Algebraic classification of RNV spacetimes. For each subtype, the necessary and sufficient conditions are presented for the case of Einstein spacetimes and for general form of the Ricci tensor.

Type	Subtype	Condition	
		general Ricci tensor	Einstein spacetimes
II	II(a)	$\frac{R}{n-1} + (n-4)DL_{11} = 0$	$\Lambda = 0$
	II(b)	$R_{ij} - \frac{R}{n-2}\delta_{ij} + \frac{2DL_{11}}{n-2}\delta_{ij} = 0$	identically satisfied
	II(c)	eq. (B.15)	$C_{ijkl} = \frac{8\Lambda}{(n-1)(n-2)(n-3)}\delta_{i[k}\delta_{l]j}$
	II(d)	identically satisfied	identically satisfied
III	III(a)	$\text{II(abc)} \wedge \delta_i L_{11} - \frac{1}{n-2}R_{1i} = 0$	$\text{II(ac)} \wedge \delta_i L_{11} = 0$
	III(b)	$\text{II(abc)} \wedge \text{eq. (B.16)}$	$\text{II(ac)} \wedge C_{1ijk}C_{1ijk} = \frac{2}{n-3}(\delta_i L_{11})^2$
N		III(ab)	III(ab)

Table B2. Algebraic classification of pp -waves (CCNV spacetimes). The necessary and sufficient conditions determining the corresponding subtype are given for general Ricci tensor and for the Ricci-flat case.

Type	Subtype	Condition	
		general Ricci tensor	Ricci-flat
II	II(a)	$R = 0$	identically satisfied
	II(b)	$R_{ij} - \frac{R}{n-2}\delta_{ij} = 0$	identically satisfied
	II(c)	eq. (B.15) with $L_{11} = 0$	$C_{ijkl} = 0$
	II(d)	identically satisfied	identically satisfied
III	III(a)	$\text{II(abc)} \wedge R_{1i} = 0$	II(c)
	III(b)	$\text{II(abc)} \wedge C_{1ijk}C_{1ijk} = \frac{2}{(n-2)(n-3)}R_{1i}R_{1i}$	$\text{II(c)} \wedge C_{1ijk}C_{1ijk} = 0$
N		III(ab)	III(b)

In the case of pp -waves, it follows that $R_{00} = R_{0i} = R_{01} = 0$ and the conditions determining the corresponding subtypes further simplify, see table B2. In general, pp -waves are of type II(d). Since $R_{01} = 0$, Einstein pp -waves are necessarily Ricci-flat and these are of type II(abd).

Some properties of the above-mentioned subclasses of Kundt spacetimes determined by the geometry of the congruence of non-expanding, non-shearing, and non-twisting null geodesics ℓ are summarized in table B3.

Table B3. Classification of Kundt spacetimes according to the geometry of the geodetic non-expanding, non-shearing, and non-twisting null vector field ℓ . For each subclass, the corresponding holonomy group and admissible Weyl type are presented. The Ricci rotation coefficients L_{11} and L_{1i} are frame components of the covariant derivative of ℓ in the canonical form (B.1). In the case $L_{1i} \neq 0$, it is always possible to set $L_{11} = 0$.

L_{11}	L_{1i}	Class	Holonomy	Ricci tensor	Weyl type
any value	non-zero	Kundt	$SO(1, n-1)$	$R_{00} = 0$	I(b)
non-zero	0	RNV	$Sim(n-2)$	$R_{00} = R_{0i} = 0$	II(d)
0	0	pp -waves	$E(n-2)$	$R_{00} = R_{0i} = R_{01} = 0$	II(d)

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